# MCS 221 Final Exam Solutions

1. (10 pts) Let $F$ be a field and $b \in F$. Prove that
$$U = \{(x_1, x_2, x_3, x_4) \in F^4 \mid x_3 = 5x_4 + b\}$$
is a subspace of $F^4$ if and only if $b = 0$.

Suppose $U$ is a subspace. In this case, $0 \in U$, so the vector $(0,0,0,0)$ must satisfy $x_3 = 5x_4 + b$, that is $0 = 0 + b$. Hence $b = 0$.

Conversely, suppose $b = 0$. Then $(0,0,0,0)$ satisfies $x_3 = 5x_4 + b$, hence $(0,0,0,0) \in U$. Now, suppose $(x_1, x_2, x_3, x_4)$ and $(y_1, y_2, y_3, y_4)$ are in $U$, so $x_3 = 5x_4$ and $y_3 = 5y_4$. Then
$$x_3 + y_3 = 5x_4 + 5y_4 = 5(x_4 + y_4).$$
Hence
$$(x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4) \in U.$$
Finally, suppose $(x_1, x_2, x_3, x_4) \in U$ and $a \in F$. Then $x_3 = 5x_4$, so $ax_3 = a(5x_4) = 5(ax_4)$, so
$$a(x_1, x_2, x_3, x_4) = (ax_1, ax_2, ax_3, ax_4) \in U.$$
Therefore $U$ is closed under addition and scalar multiplication and contains 0, hence $U$ is a subspace of $F^4$.

2. (10 pts) Suppose $U$ and $W$ are subspaces of $V$ such that $V = U \oplus W$. Suppose $u_1, \dots, u_m$ is a basis of $U$ and $w_1, \dots, w_n$ is a basis of $W$. Prove that
$$u_1, \dots, u_m, w_1, \dots, w_n$$
is a basis of $V$.

We will first show that any vector in $U+W$ is a linear combination of $u_1, \dots, u_m, w_1, \dots, w_n$. Let $v \in U + W$. So $v = u + w$ for some $u \in U$ and $w \in W$. Since $u_1, \dots, u_m$ span $U$ and $w_1, \dots, w_n$ span $W$, we can write
$$u = a_1 u_1 + \cdots + a_m u_m$$
$$w = b_1 w_1 + \cdots + b_n w_n$$
for some $a_1, \dots, a_m, b_1, \dots, b_n \in F$. Hence
$$v = a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots + b_n w_n.$$

Now, to show that $u_1, \dots, u_m, w_1, \dots, w_n$ is linearly independent, suppose
$$a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots b_n w_n = 0.$$
Then the vector
$$v = a_1 u_1 + \cdots + a_m u_m = -b_1 w_1 - \cdots - b_n w_n$$
is both in $U$ and $W$. Since $U + W$ is direct, $U \cap W = \{0\}$. So $v = 0$. Now, $a_i = 0$ for all $i$ by the linear independence of $u_1, \dots, u_m$ and $b_j = 0$ for all $j$ by the linear independence of $w_1, \dots, w_n$.

3. (10 pts) Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \dots, v_m$ is a list of vectors in $V$ such that $T(v_1), \dots, T(v_m)$ is a linearly independent list in $W$. Prove that $v_1, \dots, v_m$ is linearly independent.

Let $a_1, \dots, a_m$ be scalars such that
$$a_1 v_1 + \cdots + a_m v_m = 0.$$

Then

$$a_1T(v_1) + \cdots + a_mT(v_m) = T(a_1v_1 + \cdots + a_mv_m) = T(0) = 0.$$

Since $T(v_1), \ldots, T(v_m)$ is linearly independent, $a_1 = \cdots = a_m = 0$. Hence $v_1, \ldots, v_m$ is linearly independent.

4. (a) (3 pts) State the definition of infinite-dimensional vector space.

   A vector space $V$ is infinite-dimensional if there is no finite list of vectors in $V$ that spans $V$.

   (b) (7 pts) Give an example of an infinite-dimensional vector space. Of course, you need to prove that your example satisfies the definition in part (a).

   A good example is $\mathcal{P}(F)$ over any field $F$. See Example 2.16 in LADR for the proof that $\mathcal{P}(F)$ is infinite-dimensional.

5. (10 pts) Let

$$A = \begin{pmatrix} 2 & -1 & -2 \\ 1 & -2 & -7 \\ -3 & 2 & 5 \end{pmatrix}$$

and let $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ be the linear map $T(v) = Av$. Find $\operatorname{null}(T)$.

Since

$$\operatorname{null}(T) = \{v \in \mathbb{R}^3 \mid T(v) = 0\} = \{v \in \mathbb{R}^3 \mid Av = 0\}$$

we can find the null space by solving the homogeneous linear equation $Av = 0$. A good way to do this is by row-reducing the augmented matrix of the equation. Let

$$v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

so $Av = 0$ is the matrix equation

$$\begin{pmatrix} 2 & -1 & -2 \\ 1 & -2 & -7 \\ -3 & 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then

$$\begin{aligned}
\begin{pmatrix} 2 & -1 & -2 & 0 \\ 1 & -2 & -7 & 0 \\ -3 & 2 & 5 & 0 \end{pmatrix} &\xrightarrow{3R_2+R_3 \to R_3} \begin{pmatrix} 2 & -1 & -2 & 0 \\ 1 & -2 & -7 & 0 \\ 0 & -4 & -16 & 0 \end{pmatrix} \\
&\xrightarrow{-2R_2+R_1 \to R_1} \begin{pmatrix} 0 & 3 & 12 & 0 \\ 1 & -2 & -7 & 0 \\ 0 & -4 & -16 & 0 \end{pmatrix} \\
&\xrightarrow{\frac{1}{3}R_1 \to R_1} \begin{pmatrix} 0 & 1 & 4 & 0 \\ 1 & -2 & -7 & 0 \\ 0 & -4 & -16 & 0 \end{pmatrix}
\end{aligned}$$

$$\xrightarrow{-\frac{1}{4}R_3\to R_3} \begin{pmatrix} 0 & 1 & 4 & 0 \\ 1 & -2 & -7 & 0 \\ 0 & 1 & 4 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3-R_1\to R_3} \begin{pmatrix} 0 & 1 & 4 & 0 \\ 1 & -2 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1\leftrightarrow R_2} \begin{pmatrix} 1 & -2 & -7 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{2R_2+R_1\to R_1} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

That is

$$\begin{aligned} x_1 + x_3 = 0 &\implies x_1 = -x_3 \\ x_2 + 4x_3 = 0 &\implies x_2 = -4x_3 \end{aligned}$$

where $x_3 \in \mathbb{R}$ is a free variable, that is $x_3$ is any real number. So

$$\operatorname{null}(T) = \left\{ \begin{pmatrix} -a \\ -4a \\ a \end{pmatrix} \middle| \, a \in \mathbb{R} \right\},$$

or

$$\operatorname{null}(T) = \operatorname{span} \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix}.$$

6. (10 pts) State and prove the Fundamental Theorem of Linear Maps.

   See Theorem 3.22 in LADR.

7. **Extra credit problem.** Let $V$ and $W$ be finite-dimensional vector spaces over the field $F$, and let $T \in \mathcal{L}(V, W)$.

   (a) (5 pts) Suppose $U$ is also a vector space over $F$ and $S \in \mathcal{L}(U, V)$ such that $TS$ is surjective. Prove that $T$ must be surjective.

   That $TS$ is surjective means that for any $w \in W$ is in range$(TS)$, that is there exists a $u \in U$ such that $TS(u) = W$. Let $v = S(u)$. Then $w = T(v)$, which shows $w \in \operatorname{range}(T)$. This is true for any $w \in W$, so range$(T) = W$. Hence $T$ is surjective.

   (b) (10 pts) Prove that $T$ has a right inverse $S \in \mathcal{L}(W, V)$ such that $TS = I_W$ if and only if $T$ is surjective.

   First, suppose $T$ has a right inverse $S$. Then $TS = I_W$, which is obviously surjective. Hence $T$ is surjective by part (a).
   Now, suppose $T$ is surjective. We will construct an $S \in \mathcal{L}(W, V)$ such that $TS = I_w$. First, since $W$ is finite-dimensional, we can choose a basis $w_1, \ldots, w_n$ for $W$. Since $T$ is surjective, for each $w_i$ there is a $v_i \in V$ such that $T(v_i) = w_i$. By Theorem 3.5, there is a linear map $S : W \to V$ such that $S(w_i) = v_i$ for each $i$. We will show $TS = I_W$. Let $w$ be any vector in $W$. Then there exist $a_1, \ldots, a_n \in F$ such that $w = a_1 w_1 + \cdots + a_n w_n$.

So

$$\begin{aligned} TS(w) &= TS(a_1w_1 + \cdots + a_nw_n) \\ &= a_1TS(w_1) + \cdots + a_nTS(w_n) \\ &= a_1w_1 + \cdots + a_nw_n \\ &= w \end{aligned}$$

since $TS(w_i) = T(S(w_i)) = T(v_i) = w_i$ for each $i$. We conclude $TS = I_W$.