# MCS 221 Introduction to complex numbers

Lecture notes for MCS 221

We will define the set of complex numbers and investigate some of their algebraic properties.

**Definition 1.** Define $i$ to be a number (necessarily not real) such that $i^2 = -1$. The set of complex numbers is

$$\mathbb{C} = \{x + yi \mid x, y \in \mathbb{R}\}.$$

Define addition on $\mathbb{C}$ as

$$(x + yi) + (a + bi) = (x + a) + (y + b)i$$

and multiplication as

$$(x + yi)(a + bi) = (xa - yb) + (xb + ya)i.$$

The latter complicated-looking formula comes from using the distributive property to multiply $x + yi$ and $a + bi$:

$$\begin{aligned}(x + yi)(a + bi) &= xa + x(bi) + (yi)a + (yi)(bi)\\ &= xa + (xb)i + (ya)i + (yb)i^2\\ &= xa + (xb + ya)i - yb\\ &= (xa - yb) + (xb + ya)i.\end{aligned}$$

where we also used commutativity and associativity of multiplication to rearrange the terms. Of course, there is no good reason to assume that multiplication has the same commutative, associative, and distributive properties on $\mathbb{C}$ as it has on $\mathbb{R}$. It is more that we wish it should and define addition and multiplication accordingly. In fact, it remains to be verified that these common properties of addition and multiplication are still satisfied on $\mathbb{C}$.

But first, let us convince ourselves that addition and multiplication are operations on $\mathbb{C}$, that is the sum and product of two complex numbers are still in the set $\mathbb{C}$. This is quite obvious from the above definitions. If $x + yi$ and $a + bi$ are complex numbers, then $x$, $y$, $a$, and $b$ must be real numbers. So $x + a$ and $y + b$ are also real numbers and hence

$$(x + yi) + (a + bi) = (x + a) + (y + b)i \in \mathbb{C}.$$

Verifying that $(x + yi)(a + bi) \in \mathbb{C}$ is similar. I will leave it to you to do it.

Now, back to commutativity and associativity. Let $x + yi$ and $a + bi$ be complex numbers. Then $x, y, a, b \in \mathbb{R}$. Now

$$\begin{aligned}(x + yi) + (a + bi) &= (x + a) + (y + b)i\\ (a + bi) + (x + yi) &= (a + x) + (b + y)i\end{aligned}$$

and these are equal to each other since $x + a = a + x$ and $y + b = b + y$ be the commutative property of addition on real numbers. I will leave it to you to verify that addition of complex numbers is also associative and that multiplication is both commutative and associative.

Similarly, you can verify by direct computation that multiplication is distributive over addition, that is if $x + yi$, $a + bi$, and $c + di$ are complex numbers then

$$(x + yi)\big((a + bi) + (c + di)\big) = (x + yi)(a + bi) + (x + yi)(c + di).$$

By commutativity of multiplication it also follows that

$$\big((a + bi) + (c + di)\big)(x + yi) = (a + bi)(x + yi) + (c + di)(x + yi).$$

It is similarly easy to check that $0 + 0i$ works as an additive identity and $1 + 0i$ works as a multiplicative identity. (Go ahead and do it.) In fact, since $0i = 0$, the additive identity and the multiplicative identity are the usual numbers 0 and 1 you know and love in $\mathbb{R}$. This is a good time to point out that any real number $x$ is also a complex number, as you can always write it as $x = x + 0i$. This shows that $\mathbb{R} \subset \mathbb{C}$.

That each complex number has an additive inverse is also an easy fact to verify. The obvious candidate for $-(x+yi)$ is $-x+(-y)i$. Can you see that

$$(x+yi) + \big(-x+(-y)i\big) = 0 \text{ and } \big(-x+(-y)i\big) + (x+yi) = 0?$$

What about multiplicative inverses? Those are a little more difficult to obtain. Let $z = x+yi$ be a complex number. Then $w = a+bi$ is a multiplicative inverse of $z$ if

$$1 = zw = (x+yi)(a+bi) = (xa-yb) + (xb+ya)i.$$

This implies

$$\begin{aligned} xa - yb &= 1 \\ xb + ya &= 0 \end{aligned}$$

Multiply the first equation by $x$ and the second by $y$ to get

$$\begin{aligned} x^2a - xyb &= x \\ yxb + y^2a &= 0 \end{aligned}$$

Now add them:

$$x + 0 = x^2a - xyb + yxb + y^2a = (x^2+y^2)a implies a = \frac{x}{x^2+y^2},$$

as long as $x^2+y^2 \neq 0$. Similarly, multiplying the first equation by $-y$ and the second by $x$ and then adding them yields

$$(y^2+x^2)b = -y \implies b = \frac{-y}{x^2+y^2},$$

as long as $x^2+y^2 \neq 0$. Notice that the only way $x^2+y^2 = 0$ is if $x = y = 0$, which would make $z = 0$. So if $z \neq 0$,

$$w = \frac{x}{x^2+y^2} + \frac{-y}{x^2+y^2}i$$

is a multiplicative inverse of $z$. This means that every nonzero element $z$ of $\mathbb{C}$ has a multiplicative inverse. We will refer to

$$\frac{x}{x^2+y^2} + \frac{-y}{x^2+y^2}i$$

as $z^{-1}$.

Finally, we need to verify $0 \neq 1$ in $\mathbb{C}$. To do so, we need to define what it means for two complex numbers $z = x+yi$ and $w = a+bi$ to be equal. To motivate the definition, note that

$$z = w \implies x+yi = a+bi \implies x-a = bi-yi = (b-y)i.$$

Now, if $b-y \neq 0$, then

$$i = \frac{x-a}{b-y} \in \mathbb{R}.$$

But we defined $i$ to be a number such that $i^2 = -1$ and no real number can possibly satisfy this equation. So $i$ cannot be a real number and therefore $b-y$ must be 0. So $y = b$. But then

$$x-a = bi-yi = (b-y)i = 0i = 0 \implies x = a.$$

Technically, we made a few leaps of faith in the above argument–if you don't see what they are, that's alright–, so instead of accepting the result as a fact we proved, I will state it as a definition:

**Definition 2.** If $z = x+yi$ and $w = a+bi$ are complex numbers, we say $z = w$ if $x = a$ and $y = b$.

Now it should be clear that

$$0 = 0 + 0i \neq 1 + 0i = 1.$$

We can conclude that $\mathbb{C}$ with addition and multiplication as defined above is a field. Note that as usual, you can define subtraction in $\mathbb{C}$ by

$$z - w = z + (-w)$$

for any $z, w \in \mathbb{C}$ and division by

$$\frac{z}{w} = zw^{-1}$$

for any $z, w \in \mathbb{C}$ such that $w \neq 0$.

It it time to introduce some handy terminology:

**Definition 3.** If $z = x + yi$ is a complex number, then the *real part* of $z$ is $\mathrm{Re}(z) = x$ and the *imaginary part* of $z$ is $\mathrm{Im}(z) = y$. If $\mathrm{Re}(z) = 0$ then we call $z$ an *imaginary number*.

Note that the imaginary part of a complex number is a real number.

**Definition 4.** The *(complex) conjugate* of a complex number $z = x + yi$ is

$$\overline{z} = x - yi.$$

For example, $\overline{3 + 4i} = 3 - 4i$. To relate this to something you are already familiar with, recall that you learned in algebra class to rationalize a fraction whose denominator had a square root in it by multiplying the numerator and the denominator by the conjugate, such as

$$\frac{3}{5+\sqrt{8}} = \frac{3}{5+\sqrt{8}}\,\frac{5-\sqrt{8}}{5-\sqrt{8}} = \frac{3(5-\sqrt{8})}{5^2-\sqrt{8}^2} = \frac{5-\sqrt{8}}{5-\sqrt{8}} = \frac{3(5-\sqrt{8})}{25-8} = \frac{3(5-\sqrt{8})}{17}.$$

Now, if you think of $i$ as $\sqrt{-1}$ then it should make sense that we call $\overline{z} = x - yi = x - y\sqrt{-1}$ the conjugate of $z = x + yi = x + y\sqrt{-1}$.

It may not be immediately clear why we would want to define $\overline{z}$, but note that

$$z\overline{z} = (x + yi)(x - yi) = x^2 + y^2$$

is a nonnegative real number and that

$$\frac{\overline{z}}{z\overline{z}} = \frac{x - yi}{x^2 + y^2} = \frac{x}{x^2 + y^2} + \frac{-y}{x^2 + y^2}i = z^{-1}.$$

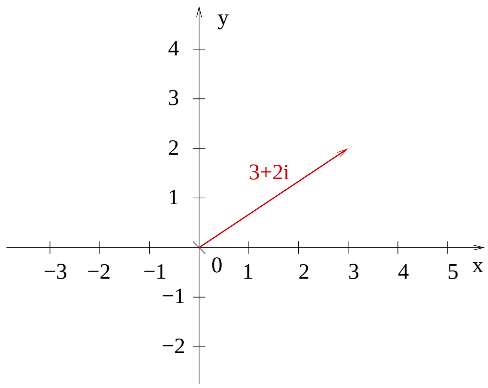
So we have an easy way or memorizing how to calculate $z^{-1}$. Fun fact: a complex number $z$ is a real number if and only if $z = \overline{z}$. Can you see why?

Here are a few useful properties of conjugation:

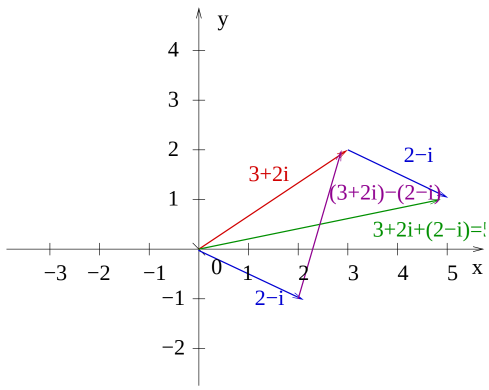
$$\begin{aligned}
\overline{w + z} &= \overline{w} + \overline{z} \\
\overline{w - z} &= \overline{w} - \overline{z} \\
\overline{wz} &= \overline{w}\,\overline{z} \\
\overline{\frac{w}{z}} &= \frac{\overline{w}}{\overline{z}} \\
\mathrm{Re}\, z &= \frac{z + \overline{z}}{2} \\
\mathrm{Im}\, z &= \frac{z - \overline{z}}{2i}
\end{aligned}$$

for any $w, z \in \mathbb{C}$. You can easily verify these.

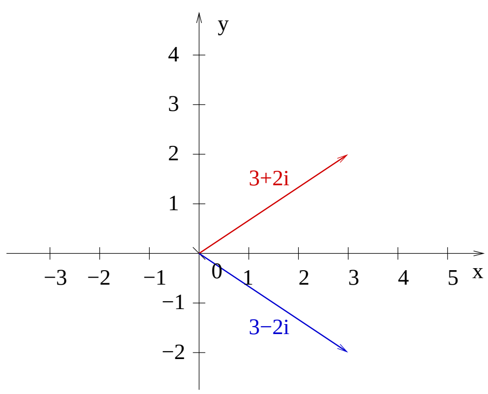
There is a nice geometric way to visualize complex numbers as points in a plane. Start with the usual 2-dimensional Euclidean plane, i.e. the $xy$-plane you know well from calculus class. Consider the $x$-axis the real number line and the $y$-axis a line that contains the imaginary numbers of the form $yi$ where $y \in \mathbb{R}$. Now the complex number $z = x + yi$ is just the point $(x, y)$ in this coordinate plane. E.g.




Notice that the real number line is part of the complex plane. Addition and subtraction of complex numbers are easy to visualize in this setting by arrows:




Multiplication and division also have geometric interpretations, but they are less obvious. I will not talk about them here. Conjugation has a nice a geometric meaning as reflection across the real axis:




Finally, notice

$$z\overline{z} = (x + yi)(x - yi) = x^2 + y^2$$

is the square of the usual Euclidean distance of $z$ from the origin. In fact, this deserves its own definition:

**Definition 5.** For a complex number $z = x + yi$, the *absolute value* or *magnitude* or *length* of $z$ is

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\overline{z}}.$$

Notice that just like the absolute value of real numbers, $|z| \geq 0$ for every $z \in \mathbb{C}$; and in fact, $|z| = 0$ only if $z = 0$. Much like on $\mathbb{R}$, the absolute value has the following useful properties:

$$\begin{aligned}
|-z| &= |z| \\
|wz| &= |w||z| \\
\left|\frac{w}{z}\right| &= \frac{|w|}{|z|} \\
|w+z| &\leq |w| + |z| \\
|w-z| &\leq |w| + |z|
\end{aligned}$$

for all $w, z \in \mathbb{C}$. The first three of these are straightforward to show by direct calculation. The third property $|w+z| \leq |w| + |z|$ is called the Triangle Inequality. You can probably see why by looking back at the diagram above that shows the geometric meaning of $w + z$. It is a little more challenging to prove, but not really difficult. In fact, the diagram of $w + z$ in the complex plane should make it intuitively clear that equality

$$|w+z| = |w| + |z|$$

holds if and only if $w$ and $z$ are arrows that point in the same direction.

That should be enough or more than enough to let you get by in MCS-221. I will say a few more things about complex numbers, just to whet your appetite for learning about them. We do not need these in MCS-221 at this point and you may ignore them if you would like.

A really neat property of complex numbers is that every polynomial of degree $n$ with complex coefficients factors as a product of linear factors. This also means that such a polynomial has exactly $n$ roots, although some of them may be equal to each other, in which case you need to count them with multiplicities. This is called the Fundamental Theorem of Algebra. Compare this with polynomials with real coefficients. Some do factor as a product of linear factors, e.g. $x^2 - 5x + 4 = (x-1)(x-4)$. Others do not, e.g. $x^2 + 1$. But over $\mathbb{C}$, even $x^2 + 1$ factors as $(x+i)(x-i)$.

You can also define many of the functions you are already familiar with over the complex numbers, such as exponential functions, logarithmic functions, and trigonometric functions. Then you can do calculus with them. These functions are related in interesting ways over the complex numbers. E.g.

$$\begin{aligned}
\sin(z) &= \frac{e^{iz} - e^{iz}}{2i} \\
\cos(z) &= \frac{e^{iz} + e^{iz}}{2} \\
\tan(z) &= \frac{e^{iz} - e^{iz}}{e^{iz} + e^{iz}}
\end{aligned}$$

These make it quite easy to understand and prove standard formulas from calculus, such as

$$\frac{d}{dz}\sin(z) = \cos(z) \text{ and } \frac{d}{dz}\cos(z) = -\sin(z).$$

If you find these intriguing, you may want to check out MCS-321, Elementary Theory of Complex Variables.