MCS 221 Exam 1 Solutions

1. (10 pts) Let F be any field. Define addition and scalar multiplication on $V = F^n$ as usual. Show that $\lambda(x+y) = \lambda x + \lambda y$ for all $\lambda \in F$ and all $x, y \in F^n$.

   Let $x, y \in F^n$. So $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ where $x_i, y_i \in F$ for all i. Now

$$\begin{aligned}\lambda(x+y) &= \lambda\big((x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n)\big)\\ &= \lambda(x_1+y_1, x_2+y_2, \dots, x_n+y_n)\\ &= \big(\lambda(x_1+y_1), \lambda(x_2+y_2), \dots, \lambda(x_n+y_n)\big)\end{aligned}$$

   Also

$$\begin{aligned}\lambda x + \lambda y &= \lambda(x_1, x_2, \dots, x_n) + \lambda(y_1, y_2, \dots, y_n)\\ &= (\lambda x_1, \lambda x_2, \dots, \lambda x_n) + (\lambda y_1, \lambda y_2, \dots, \lambda y_n)\\ &= (\lambda x_1 + \lambda y_1, \lambda x_2 + \lambda y_2, \dots, \lambda x_n + \lambda y_n)\end{aligned}$$

   Since multiplication in F is distributive over addition, $\lambda(x_i + y_i) = \lambda x_i + \lambda y_i$ for all i. Hence $\lambda(x+y) = \lambda x + \lambda y$.

2. (10 pts) Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbb{R}$. Define an addition and scalar multiplication on $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbb{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0,\\ 0 & \text{if } t = 0,\\ \infty & \text{if } t > 0,\end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0,\\ 0 & \text{if } t = 0,\\ -\infty & \text{if } t > 0,\end{cases}$$

$$t + \infty = \infty + t = \infty, \qquad t + (-\infty) = (-\infty) + t = -\infty,$$

$$\infty + \infty = \infty, \qquad (-\infty) + (-\infty) = -\infty, \qquad \infty + (-\infty) = 0.$$

   Is $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over $\mathbb{R}$? Explain.

   $\mathbb{R} \cup \{\infty, -\infty\}$ is not a vector space because addition is not associative. E.g.

$$\begin{aligned}(1+\infty) + (-\infty) &= \infty + (-\infty) = 0\\ 1 + \big(\infty) + (-\infty)\big) &= 1 + 0 = 1,\end{aligned}$$

   and these are not equal.

   In fact, it is interesting to note that $\mathbb{R} \cup \{\infty, -\infty\}$ with the addition and scalar multiplication defined above satisfies all of the other properties of a vector space.

3. (5 pts each) Let $V = \mathbb{R}^2$ over the field $\mathbb{R}$. Are the following sets subspaces of V?
   (a) $U = \{(x, y) \in \mathbb{R}^2 \mid x^2 = y^2\}$

   U is not a subspace because it is not closed under addition. For example, $u = (1, 1)$ and $v = (-1, 1)$ are both in U, since $1^2 = 1^2$ and $(-1)^2 = 1^2$. But $u + v = (0, 2)$ is not in U because $0^2 \neq 2^2$.

   (b) $U = \{(x, y) \in \mathbb{R}^2 \mid xy \geq 0\}$

   This is not a subspace either because it is not closed under addition. For example, $u = (2, 2)$ and $v = (-1, -3)$ are both in U, since $2 \cdot 2 = 4 \geq 0$ and $(-1)(-3) = 3 \geq 0$. But $u + v = (1, -1)$ is not in U because $1(-1) = -1 < 0$.

4. Let V be a vector space. Let $U_1, U_2, \ldots, U_n$ be subspaces of V.
   (a) (3 pts) State the definition of the sum $U_1 + U_2 + \cdots + U_n$.

   This is Definition 1.36 in your textbook:

   $$U_1 + U_2 + \cdots + U_n = \{u_1 + u_2 + \cdots + u_n \mid u_1 \in U_1, u_2 \in U_2, \ldots, u_n \in U_n\}.$$

   (b) (7 pts) Prove that $U_1 + U_2 + \cdots + U_n$ is a subspace of V.

   This is part of Theorem 1.39 in your textbook. Let $U = U_1 + U_2 + \cdots + U_n$. First, notice that if $u \in U$, then $u = u_1 + u_2 + \cdots + u_n$ for some $u_i \in U_i$. Since u_i is also in V for all i, and V is a vector space, hence closed under addition, $u = u_1 + u_2 + \cdots + u_n \in V$. This is true for all $u \in U$. So $U \subseteq V$.
   Now, $0 = 0 + 0 + \cdots + 0 \in U_1 + U_2 + \cdots + U_n$. Let $u, v \in U$. Then

   $$u = u_1 + u_2 + \cdots + u_n, \qquad v = v_1 + v_2 + \cdots + v_n$$

   for some $u_i, v_i \in U_i$. Hence

   $$\begin{aligned} u + v &= (u_1 + u_2 + \cdots + u_n) + (v_1 + v_2 + \cdots + v_n) \\ &= (u_1 + v_1) + (u_2 + v_2) + \cdots + (u_n + v_n) \end{aligned}$$

   where we used associativity and commutativity of addition to rearrange the sum. Since each U_i is a subspace, it is closed under addition, so $u_i + v_i \in U_i$. This shows $u + v \in U$. Now let $\lambda \in F$. Then

   $$\lambda u = \lambda(u_1 + u_2 + \cdots + u_n) = \lambda u_1 + \lambda u_2 + \cdots + \lambda u_n$$

   by distributivity. Each $\lambda u_i \in U_i$ by closure under scalar multiplication. Therefore $\lambda u \in U$. By Theorem 1.34, U is a subspace of V.

5. **Extra credit problem.**
   (a) (6 pts) Let F be a field. Show that if x and y are nonzero elements of F then $xy \neq 0$.

   By way of contradiction, suppose $xy = 0$. Since $x \neq 0$, it has a multiplicative inverse x^{-1} in F. Multiply both sides of $xy = 0$ by x^{-1} and use associativity of multiplication to get

   $$x^{-1}0 = x^{-1}(xy) = (x^{-1}x)y = 1y = y.$$

   In a field, much like in a vector space, $x^{-1}0 = 0$. We can show this the same way as in a vector space:

   $$x^{-1}0 = x^{-1}(0 + 0) = x^{-1}0 + x^{-1}0,$$

   and now we can add the additive inverse of $x^{-1}0$, call it z, to both sides:

   $$0 = x^{-1}0 + z = (x^{-1}0 + x^{-1}0) + z = x^{-1}0 + (x^{-1}0 + z) = x^{-1}0 + 0 = x^{-1}0.$$

   Hence $y = x^{-1}0 = 0$, which contradicts $y \neq 0$. Therefore xy cannot be 0.

   Note that if it feels like these are the same arguments we gave in class for proving Theorem 1.30 and you may have given on your homework for exercise 1.B.2, that is no coincidence. They are the same arguments. This is because any field is a vector space over itself, using the same field addition and multiplication as the vector space addition and scalar multiplication. So Theorem 1.30 applied to the vector space F over the field F does in fact show that any element times 0 is 0. We could have simply cited Theorem 1.30 here instead of giving the argument above. In the specific context of F over F, exercise 1.B.2 says exactly the same thing as this problem on this exam.

(b) (4 pts) Use the result in part (a) to prove that $\mathbb{Z}_n$ (the integers modulo n with the usual addition and multiplication) cannot be a field if n is a composite number.

Let n be a composite number. Then $n = km$ for some integers $1 < k, m < n$. Notice that k and m cannot be divisible by n (because $1 < k, m < n$), so $\overline{k}$ and $\overline{m}$ are not equal to $\overline{0}$ in $\mathbb{Z}_n$. But $\overline{k}\,\overline{m} = \overline{n} = \overline{0}$. We just showed in part (a) that this cannot happen in a field, therefore $\mathbb{Z}_n$ cannot be a field.