# MCS 221 Exam 2 Solutions

1. (10 pts) A function $f : \mathbb{R} \to \mathbb{R}$ is called *even* if
$$f(-x) = f(x)$$
for all $x \in \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is called *odd* if
$$f(-x) = -f(x)$$
for all $x \in \mathbb{R}$. Let $U_e$ denote the subspace of even functions and let $U_o$ denote the subspace of odd functions in $\mathbb{R}^{\mathbb{R}}$. Show that $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$. You do not need to show that $U_e$ and $U_o$ are subspaces of $\mathbb{R}^{\mathbb{R}}$; you may assume that we already know this..

Let $f$ be any function in $V$. Let $g(x) = f(x) + f(-x)$. Notice that
$$g(-x) = f(-x) + f(-(-x)) = f(-x) + f(x) = g(x)$$
for any $x \in \mathbb{R}$, so $g$ is an even function. Let $h(x) = f(x) - f(-x)$. Then
$$h(-x) = f(-x) - f(-(-x)) = f(-x) - f(x) = -\big(f(x) - f(-x)\big) = -h(x)$$
for any $x \in \mathbb{R}$, so $h$ is an odd function. Now let $u(x) = \frac{f(x)+f(-x)}{2}$ and $w(x) = \frac{f(x)-f(-x)}{2}$. Then
$$u(-x) = \frac{f(-x) + f(-(-x))}{2} = \frac{f(-x) + f(x)}{2} = u(x)$$
$$w(-x) = \frac{f(-x) - f(-(-x))}{2} = \frac{f(-x) - f(x)}{2} = w(x)$$
for any $x \in \mathbb{R}$. So $u \in U_e$ and $w \in U_o$. Notice $u + w = f$ since
$$u(x) + w(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = \frac{2f(x)}{2} = f(x),$$
and hence $f \in U_e + U_o$. Since this is true for any $f \in V$, we have shown that $\mathbb{R}^{\mathbb{R}} = U_e + U_o$.

Now, we will show $U_e \cap U_o = \{0\}$ and conclude that $U_e + U_o$ is direct by Theorem 1.45. Let $f \in U_e \cap U_o$. Then $f$ is both even and odd. Hence $f(-x) = f(x)$ and $f(-x) = -f(x)$ for any $x$. Therefore
$$f(x) = -f(x) \implies 2f(x) = 0 \implies f(x) = 0.$$
Hence $f$ must indeed be the 0 function.

2. (10 pts) Let $V$ be a vector space. Suppose that $v_1, \ldots, v_m$ is linearly independent in $V$ and $w \in V$. Show that $v_1, \ldots, v_m, w$ is linearly independent if and only if $w \notin \operatorname{span}(v_1, \ldots, v_m)$.

We will show the logically equivalent statement that $v_1, \ldots, v_m, w$ is linearly dependent if and only if $w \in \operatorname{span}(v_1, \ldots, v_m)$.

First, suppose $w \in \operatorname{span}(v_1, \ldots, v_m)$. Then
$$w = a_1 v_1 + \cdots + a_m v_m$$
for some scalars $a_i$. Therefore
$$0 = a_1 v_1 + \cdots + a_m v_m + (-1)w.$$
Since not all scalars on the right-hand side are 0, $v_1, \ldots, v_m, w$ is linearly dependent.

Now, suppose $v_1, \ldots, v_m, w$ is linearly dependent. By the Linear Dependence Lemma, there must be a vector in the list that is in the span of the preceding ones. This cannot be one of the $v_i$ because $v_1, \ldots, v_m$ is linearly independent. Hence $w$ must be in the spand of $v_1, \ldots, v_m$.

3. (a) (3 pts) Define what it means for a vector space to be infinite-dimensional.

   A vector space $V$ is infinite-dimensional if there is no finite list of vectors in $V$ that spans $V$.

   (b) (7 pts) Give an example of a vector space that is infinite-dimensional. Prove that your example is in fact infinite-dimensional.

   An example is the vector space of polynomials $F[x]$ over any field $F$. See Example 2.16 in LADR for the proof that $\mathcal{P}(F)$ is infinite-dimensional.

4. (10 pts) Let $V$ be a finite-dimensional vector space and let $U$ be a subspace of $V$. We proved that there exists a subspace $W$ of $V$ such that $V = U \oplus W$ by choosing a basis $u_1, \ldots, u_m$ of $U$ (we can do this because as a subspace of a finite-dimensional vector space, $U$ must be finite-dimensional, and hence it has a finite basis), extending $u_1, \ldots, u_m$ to a basis $u_1, \ldots, u_m, w_1, \ldots, w_n$ by choosing

$$\begin{aligned}
&w_1 \notin \operatorname{span}(u_1, \ldots, u_m) \\
&w_2 \notin \operatorname{span}(u_1, \ldots, u_m, w_1) \\
&\quad\vdots \\
&w_n \notin \operatorname{span}(u_1, \ldots, u_m, w_1, \ldots, w_{n-1})
\end{aligned}$$

until we could no longer find a vector in $V$ that is not already in the span of the extended list, and finally setting $W = \operatorname{span}(w_1, \ldots, w_n)$.

Show that the subspace $W$ we got this way does in fact satisfy $V = U \oplus W$.

First, we will show $V = U + W$. Let $v \in V$. Since $u_1, \ldots, u_m, w_1, \ldots, w_n$ is a basis of $V$,

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots + b_n w_n$$

for some scalars $a_i$ and $b_i$. Let

$$\begin{aligned}
u &= a_1 u_1 + \cdots + a_m u_m \in U \\
w &= b_1 w_1 + \cdots + b_n w_n \in W
\end{aligned}$$

Then $v = u + w$ and hence $v \in U + W$. This shows $V = U + W$.

To show that the sum is direct, we will prove $U \cap W = \{0\}$. Let $v \in U \cap W$. Then $v \in U$ and hence

$$v = a_1 u_1 + \cdots + a_m u_m$$

for some scalars $a_i$. But $v \in W$ as well, so

$$v = b_1 w_1 + \cdots b_n w_n$$

for some scalars $b_i$. Now

$$0 = v - v = a_1 u_1 + \cdots + a_m u_m - b_1 w_1 - \cdots - b_n w_n.$$

Since $u_1, \ldots, u_m, w_1, \ldots, w_n$ is linearly independent, all the $a_i$ and $b_i$ must be 0. Therefore $v = 0$.

5. (10 pts) **Extra credit problem.** Let $V$ be a vector space and let $U_1, \ldots, U_n$ be subspaces of $V$ such that the sum $U_1 + \cdots + U_n$ is direct. Suppose that each $U_i$ has a finite basis $u_{i1}, u_{i2}, \ldots, u_{im_i}$. Prove that the combined list

$$u_{11}, u_{12}, \ldots, u_{1m_1}, u_{21}, u_{22}, \ldots, u_{2m_2}, \ldots, u_{n1}, u_{n2}, \ldots, u_{nm_n}$$

is a basis of $U_1 \oplus \cdots \oplus U_n$

First, we will show that any $v \in U_1 + \cdots + U_n$ is a linear combination of the combined list of $u_{ij}$. Since $v \in U_1 + \cdots + U_n$,

$$v = v_1 + v_2 + \cdots v_n$$

for some $v_i \in U_i$. Now, each such $v_i$ is a linear combination

$$v_i = a_{i1}u_{i1} + \cdots + a_{im_i}u_{im_i}$$

for some scalars $a_{i1}, \ldots, a_{im_i}$. Therefore

$$v = a_{11}u_{11} + \cdots + a_{1m_1}u_{1m_1} + \cdots + a_{n1}u_{n1} + \cdots + a_{nm_n}u_{nm_n},$$

so $v$ is in the span of the combined list of $u_{ij}$.

Now, we will show that the combined list of $u_{ij}$ is linearly independent. Let

$$0 = a_{11}u_{11} + \cdots + a_{1m_1}u_{1m_1} + \cdots + a_{n1}u_{n1} + \cdots + a_{nm_n}u_{nm_n}.$$

Let $v_i = a_{i1}u_{i1} + \cdots + a_{im_i}u_{im_i} \in U_i$. Then

$$0 = v_1 + \cdots v_n.$$

Since $U_1 \oplus \cdots \oplus U_n$ is direct, the only way to express 0 as a sum $0 = v_1 + \cdots v_n$ with each $v_i \in U_i$ is that each $v_i = 0$. Now,

$$0 = v_i = a_{i1}u_{i1} + \cdots + a_{im_i}u_{im_i}.$$

Since $u_{i1}, u_{i2}, \ldots, u_{im_i}$ is a basis of $U_i$, and hence linearly independent, $a_{ij} = 0$ for all $j = 1, \ldots, m_i$. Therefore all of the $a_{ij}$ above must be 0, which proves that the combined list of $u_{ij}$ is linearly independent.