MCS 221 Final Exam Solutions

May 24, 2019

1. (10 pts) Let V be a vector space. Is the operation of addition on the subspaces of V associative? In other words, if U_1, U_2, U_3 are subspaces of V, is

$$(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)?$$

(Hint: remember that two sets A and B are equal if $A \subseteq B$ and $B \subseteq A$.)

Yes, it is. We will show that $(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)$. First, let u be any element in $(U_1 + U_2) + U_3$. Then $u = (u_1 + u_2) + u_3$ for some $u_1 \in U_1$, $u_2 \in U_2$, and $u_3 \in U_3$. Hence

$$u = (u_1 + u_2) + u_3 = u_1 + (u_2 + u_3) \in U_1 + (U_2 + U_3).$$

Hence $(U_1 + U_2) + U_3 \subseteq U_1 + (U_2 + U_3)$. By an analogous argument, $U_1 + (U_2 + U_3) \subseteq (U_1 + U_2) + U_3$.

2. (10 pts) Let V be a vector space. Suppose $U_1, \dots, U_m$ are finite-dimensional subspaces of V such that $U_1 + \cdots + U_m$ is a direct sum. Prove that $U = U_1 \oplus \cdots \oplus U_m$ is finite-dimensional and

$$\dim(U_1 \oplus \cdots \oplus U_m) = \dim(U_1) + \cdots + \dim(U_m).$$

Note that this is almost the same problem as the extra credit problem on our second midterm. In fact, by that problem, combining bases of $U_1, \dots, U_m$ results in a basis of $U_1 \oplus \cdots \oplus U_m$. Hence the dimension of $U_1 \oplus \cdots \oplus U_m$ is the sum of the dimensions of $U_1, \dots, U_m$. It follows that $U_1 \oplus \cdots \oplus U_m$ is finite-dimensional. For the sake of completeness, here is the whole argument, which includes the proof of the extra credit problem on the second midterm.

Since each U_i is finite-dimensional, it has a basis $u_{i1}, u_{i2}, \dots, u_{in_i}$ where $n_i = \dim(U_i)$. We will prove that the combined list

$$u_{11}, u_{12}, \dots, u_{1n_1}, u_{21}, u_{22}, \dots, u_{2n_2}, \dots, u_{m1}, u_{m2}, \dots, u_{mn_m}$$

is a basis of $U_1 \oplus \cdots \oplus U_m$.

First, we will show that any $v \in U_1 + \cdots + U_m$ is a linear combination of the combined list of u_{ij}. Since $v \in U_1 + \cdots + U_m$,

$$v = v_1 + v_2 + \cdots v_m$$

for some $v_i \in U_i$. Now, each such v_i is a linear combination

$$v_i = a_{i1}u_{i1} + \cdots + a_{in_i}u_{in_i}$$

for some scalars $a_{i1}, \dots, a_{in_i}$. Therefore

$$v = a_{11}u_{11} + \cdots + a_{1n_1}u_{1n_1} + \cdots + a_{m1}u_{m1} + \cdots + a_{mn_m}u_{mn_m},$$

so v is in the span of the combined list of u_{ij}.

Now, we will show that the combined list of u_{ij} is linearly independent. Let

$$0 = a_{11}u_{11} + \cdots + a_{1n_1}u_{1n_1} + \cdots + a_{m1}u_{m1} + \cdots + a_{mn_m}u_{mn_m}.$$

Let $v_i = a_{i1}u_{i1} + \cdots + a_{in_i}u_{in_i} \in U_i$. Then

$$0 = v_1 + \cdots v_m.$$

Since $U_1 \oplus \cdots \oplus U_m$ is direct, the only way to express 0 as a sum $0 = v_1 + \cdots v_m$ with each $v_i \in U_i$ is that each $v_i = 0$. Now,

$$0 = v_i = a_{i1}u_{i1} + \cdots + a_{in_i}u_{in_i}.$$

Since $u_{i1}, u_{i2}, \ldots, u_{in_i}$ is a basis of U_i, and hence linearly independent, $a_{ij} = 0$ for all $j = 1, \ldots, n_i$. Therefore all of the a_{ij} above must be 0, which proves that the combined list of u_{ij} is linearly independent.

Now, the list

$$u_{11}, u_{12}, \ldots, u_{1n_1}, u_{21}, u_{22}, \ldots, u_{2n_2}, \ldots, u_{m1}, u_{m2}, \ldots, u_{mn_m}$$

has $n_1 + n_2 + \cdots + n_m$ elements, so

$$\dim(U_1 \oplus \cdots \oplus U_m) = n_1 + n_2 + \cdots + n_m = \dim(U_1) + \cdots + \dim(U_m).$$

Many of you tried to give a proof by induction. This works but there are two important parts of the argument one needs to be careful about. I will point them out. Here is how such an argument would go.

For the base case, note that if $m = 1$ then the statement is $\dim(U_1) = \dim(U_1)$, which is obviously true.

For the inductive hypothesis, suppose that if $U_1, \ldots, U_k$ are finite-dimensional subspaces of V such that $U_1 + \cdots + U_k$ is a direct sum then

$$\dim(U_1 \oplus \cdots \oplus U_k) = \dim(U_1) + \cdots + \dim(U_k).$$

Now, let $U_1, \ldots, U_{k+1}$ be finite-dimensional subspaces of V such that $U_1 + \cdots + U_{k+1}$ is direct. Let $W = U_1 + \cdots + U_k$. Then

$$U_1 + \cdots + U_{k+1} = (U_1 + \cdots + U_k) + U_{k+1} = W + U_{k+1}$$

by problem 1 on this exam. The first subtle part of this argument is that we need to prove that $W + U_{k+1}$ is a direct sum. We will show that if $w \in W$ and $u_{k+1} \in U_{k+1}$ are such that $w + u_{k+1} = 0$ then $w = u_{k+1} = 0$, and hence $W + U_{k+1}$ is direct by Theorem 1.44. Since $w \in W = U_1 + \cdots + U_k$, $w = u_1 + \cdots + u_k$ for some $u_i \in U_i$. Then

$$0 = w + u_{k+1} = (u_1 + \cdots + u_k) + u_{k+1} = u_1 + \cdots + u_{k+1}.$$

But $U_1 + \cdots + U_{k+1}$ is direct, so $u_1 = \cdots = u_{k+1} = 0$, which implies $w = 0$ in turn.

Since $W + U_{k+1}$ is direct, $W \cap U_{k+1} = \{0\}$. By Theorem 2.43,

$$\begin{aligned}
\dim(U_1 + \cdots + U_{k+1}) &= \dim(W + Uk + 1) \\
&= \dim(W) + \dim(U_{k+1}) - \dim(W \cap U_{k+1}) \\
&= \dim(W) + \dim(U_{k+1}).
\end{aligned}$$

We are now ready to use the inductive hypothesis to replace $\dim(W)$ by $\dim(U_1) + \cdots + \dim(U_k)$ in this equation. But for that, we need to know that $W = U_1 + \cdots + U_k$ is a direct sum. We do know that $U_1 + \cdots + U_{k+1}$ is a direct sum. We will show that it follows that $U_1 + \cdots + U_k$ is also direct. Let $u_i \in U_i$ for $i = 1, \ldots, k$ such that $0 = u_1 + \cdots + u_k$. Then

$$0 = u_1 + \cdots + u_k + \underbrace{0}_{\in U_{k+1}} \in U_1 + \cdots + U_{k+1}$$

and since $U_1 + \cdots + U_{k+1}$ is direct, the only way to write 0 as a sum of vectors from each U_i is if all those vectors are 0 (by Theorem 1.44). Hence $u_1 = \cdots = u_k = 0$. Therefore $U_1 + \cdots + U_k$ is also a direct sum (by Theorem 1.44 once again). Now, by the inductive hypothesis,

$$\begin{aligned}
\dim(U_1 + \cdots + U_{k+1}) &= \dim(W) + \dim(U_{k+1}) \\
&= \dim(U_1) + \cdots + \dim(U_k) + \dim(U_{k+1}),
\end{aligned}$$

which is what we wanted to prove.

3. (10 pts) Suppose $S_1, \ldots, S_n$ are injective linear maps such that $S_1 S_2 \cdots S_n$ makes sense. Prove that $S_1 S_2 \cdots S_n$ is injective.

Let v be such that $S_1, \ldots, S_n(v) = 0$. We will show that $v = 0$. So

$$0 = S_1 \cdots S_n(v) = S_1\big(S_2 \cdots S_n(v)\big).$$

Since S_1 is injective, its null space is $\{0\}$, so $S_2 \cdots S_n(v) = 0$. Now

$$0 = S_2 \cdots S_n(v) = S_2\big(S_3 \cdots S_n(v)\big).$$

Since S_2 is injective, its null space is $\{0\}$, so $S_3 \cdots S_n(v) = 0$. And so on until we find that $v = 0$. This shows 0 is the only vector in the null space of $S_1 \cdots S_n$, so $S_1 \cdots S_n$ is injective.

4. (10 pts) Find the inverse of the matrix

$$A = \begin{pmatrix} 2 & 5 & 3 \\ 1 & 3 & 0 \\ -4 & -15 & 8 \end{pmatrix}$$

using row reduction.

$$\begin{pmatrix} 2 & 5 & 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 1 & 0 \\ -4 & -15 & 8 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 3 & 0 & 0 & 1 & 0 \\ 2 & 5 & 3 & 1 & 0 & 0 \\ -4 & -15 & 8 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 - 2R_1 \to R_2} \begin{pmatrix} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ -4 & -15 & 8 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_3 + 4R_1 \to R_2} \begin{pmatrix} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ 0 & -3 & 8 & 0 & 4 & 1 \end{pmatrix}$$

$$\xrightarrow{-R_2 \to R_2} \begin{pmatrix} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 \\ 0 & -3 & 8 & 0 & 4 & 1 \end{pmatrix}$$

$$\xrightarrow{3R_2 + R_3 \to R_3} \begin{pmatrix} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 \\ 0 & 0 & -1 & -3 & 10 & 1 \end{pmatrix}$$

$$\xrightarrow{-R_3 \to R_3} \begin{pmatrix} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 \\ 0 & 0 & 1 & 3 & -10 & -1 \end{pmatrix}$$

$$\xrightarrow{R_2 + 3R_3 \to R_2} \begin{pmatrix} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 8 & -28 & -3 \\ 0 & 0 & 1 & 3 & -10 & -1 \end{pmatrix}$$

$$\xrightarrow{R_1 - 3R_2 \to R_1} \begin{pmatrix} 1 & 0 & 0 & -24 & 85 & 9 \\ 0 & 1 & 0 & 8 & -28 & -3 \\ 0 & 0 & 1 & 3 & -10 & -1 \end{pmatrix}$$

So

$$A^{-1} = \begin{pmatrix} -24 & 85 & 9 \\ 9 & -28 & -3 \\ 3 & -10 & -1 \end{pmatrix}$$

5. (10 pts) Let F be a field. Prove that a matrix $A \in M_{n\times n}(F)$ is nonsingular if and only if it is row-equivalent to the identity matrix $I_n \in M_{n\times n}(F)$.

   This is Theorem NMRRI on p. 68 of Beezer. We proved it in class as part of NME1.

   First suppose A is row-equivalent to I_n. Then the augmented matrix $(A|0)$ is row-equivalent to the augmented matrix $(I_n|0)$. Row-equivalent augmented matrices correspond to equivalent systems of linear equations (by Theorem REMES in Beezer). So $Ax = 0$ has exactly the same solutions as $I_n x = 0$. Since $I_n x = x$, the only solution of $I_n x = 0$ is $x = 0$. Hence the only solution of $Ax = 0$ is the trivial solution $x = 0$. Therefore A is nonsingular.

   Conversely, suppose that A is nonsingular. The homogeneous linear equation $Ax = 0$ can be solved by row reducing the augmented matrix $(A|0)$ to $(B|0)$ where B is a matrix that is row-equivalent to A and is in reduced row-echelon form. Since A is nonsingular, the only solution of $Ax = 0$ is $x = 0$. Hence $x = 0$ is also the only solution of $Bx = 0$. This means all n variables $x_1, \ldots, x_n$ are determined. So B must have n pivot columns. But B has exactly n columns, so all of its columns must be pivot columns. The only way this is possible is if $B = I_n$. Hence A is row-equivalent to I_n.

6. (a) (4 pts) Let $T \in \mathcal{L}(V)$. Define what an eigenvalue of T is.

   A scalar λ is an *eigenvalue* of T if there is a nonzero $v \in V$ such that $T(v) = \lambda v$.

   (b) (6 pts) Give an example of vector space V over $\mathbb{R}$ and a linear map $T \in \mathcal{L}(V)$ that has an eigenvalue of 2 and an eigenvalue of -2.

   Let $V = \mathbb{R}^2$ and $T(x, y) = (2x, -2y)$. Then

   $$T(1, 0) = (2, 0) = 2(1, 0) \text{ and } T(0, 1) = (0, -2) = -2(0, 1).$$

   So 2 and -2 are both eigenvalues of T.

   Quite a few of you tried to construct an example with $V = \mathbb{R}$. This cannot possibly work. We proved that eigenvectors corresponding to distinct eigenvalues of T form a linearly independent list. Since 2 and -2 are distinct eigenvalues, the corresponding eigenvectors form a linearly independent list of length 2. Hence the dimension of V must be at least 2.

7. **Extra credit problem.**

   (a) (5 pts) Let F be a field. Prove that a matrix $A \in M_{n\times n}(F)$ has a left inverse if the rows of the matrix span F^n.

   Suppose that the rows of A span F^n. Then every vector in F^n is a linear combination of the rows. In particular, the vectors $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ in the standard basis of F^n can be expressed as

   $$e_i = b_{i1}A_{1*} + b_{i2}A_{2*} + \cdots + b_{in}A_{n*}$$

   where A_{k*} is the k-th row of A. Let B be the $n \times n$ matrix whose (i, j) entry is b_{ij}. Then BA is the matrix whose i-th row is exactly $b_{i1}A_{1*} + b_{i2}A_{2*} + \cdots + b_{in}A_{n*} = e_i$, so $BA = I_n$. Hence B is a left inverse of A.

   (b) (10 pts) Let V and W be vector spaces over the same field F and let T be a linear map in $\mathcal{L}(V, W)$. Recall that a linear map $S \in \mathcal{L}(W, V)$ is called a right inverse of T if TS is the identity map $I_W \in \mathcal{L}(W)$. Now suppose that V is finite-dimensional. Prove that T has a right inverse if and only if T is surjective.

First, suppose T has a right inverse $S \in \mathcal{L}(W, V)$. Let w be any element of W. Then $v = S(w)$ is a vector in V such that $T(v) = w$. So $w \in \text{range}(T)$. Hence $\text{range}(T) = W$, so T is surjective.

Now, suppose T is surjective. Since V is finite-dimensional, it has a basis $v_1, \ldots, v_n$. By the surjectivity of T, for any $w \in W$ there is a $v \in V$ such that $T(v)$. We can express v as a linear combination $v = a_1 v_1 + \cdots + a_n v_n$. Now

$$w = T(v) = T(a_1 v_1 + \cdots + a_n v_n) = a_1 T(v_1) + \cdots + a_n T(v_n).$$

This shows that any vector $w \in W$ is in the span of $T(v_1), \ldots, T(v_n)$. Since this spanning list is finite, W must be finite-dimensional. Hence we can choose a basis $w_1, \ldots, w_m$ for W. For each w_i there is a vector $v_i \in V$ such that $T(v_i) = w_i$. By Theorem 3.5, there is a linear map $S \in \mathcal{L}(W, V)$ such that $S(w_i) = v_i$ for i. We will show that $TS(w) = w$ for all $w \in W$. Let $w \in W$. Then w can be expressed as $w = a_1 w_1 + \cdots + a_m w_m$. So

$$\begin{aligned}
TS(w) &= TS(a_1 w_1 + \cdots + a_m w_m) \\
&= a_1 TS(w_1) + \cdots + a_m TS(w_m) \\
&= a_1 T(v_1) + \cdots + a_m T(v_m) \\
&= a_1 w_1 + \cdots + a_m w_m \\
&= w.
\end{aligned}$$

Therefore $TS = I_w$.