MCS 222 Exam 1 Solutions

1. (10 pts) Find the line through $(3, 1, 2)$ that intersects and is perpendicular to the line

$$\begin{aligned} x &= 1 + t \\ y &= 2 + t \\ z &= 1 + t. \end{aligned}$$

(Hint: If (x_0, y_0, z_0) is the point of intersection, find its coordinates.)

Let $P = (3, 1, 2)$ and $Q = (x_0, y_0, z_0)$ be the point where the two lines intersect. Then

$$\overrightarrow{PQ} = \vec{Q} - \vec{P} = (x_0 - 3, y_0 - 1, z_0 - 2)$$

is a direction vector of the line we are looking for, and hence must be perpendicular to $(1, 1, 1)$. It follows that

$$0 = \overrightarrow{PQ} \cdot (1, 1, 1) = (x_0 - 3) + (y_0 - 1) + (z_0 - 2) = x_0 + y_0 + z_0 - 6.$$

Since Q is on the line $(1 + t, 2 + t, 1 + t)$ there must be a $t \in \mathbb{R}$ such that

$$(x_0, y_0, z_0) = (1 + t, 2 + t, 1 + t).$$

Substituting these into the equation above, we get

$$\begin{aligned} (1 + t) + (2 + t) + (1 + t) - 6 &= 0 \\ 3t - 2 &= 0 \\ t &= \frac{2}{3} \end{aligned}$$

So the intersection point is

$$Q = \left(1 + \frac{2}{3}, 2 + \frac{2}{3}, 1 + \frac{2}{3}\right) = \left(\frac{5}{3}, \frac{8}{3}, \frac{5}{3}\right).$$

Hence Hence

$$\overrightarrow{PQ} = \left(\frac{5}{3}, \frac{8}{3}, \frac{5}{3}\right) - (3, 1, 2) = \left(-\frac{4}{3}, \frac{5}{3}, -\frac{1}{3}\right).$$

We can multiply this by 3 to make it look more friendly: $(-4, 5, -1)$. The equation of the line we want is $(3 - 4t, 1 + 5t, 2 - t)$.

Alternately, you could use orthogonal projection to find the direction vector of the perpendicular line through P, the way we did it in class.

2. (10 pts) Let $u, v, w \in \mathbb{R}^3$. Prove, without recourse to geometry, that

$$u \cdot (v \times w) = v \cdot (w \times u) = w \cdot (u \times v).$$

Note that we showed in class (and also in problem 4 on this exam) that if $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$, and $w = (w_1, w_2, w_3)$ then

$$(u \times v) \cdot w = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Hence

$$u \cdot (v \times w) = (v \times w) \cdot u = \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{vmatrix}.$$

Expanding the determinant along the first row gives

$$u \cdot (v \times w) = v_1(w_2u_3 - w_3u_2) - v_2(w_1u_3 - w_3u_1) + v_3(w_1u_2 - w_2u_1).$$

Similarly,

$$v \cdot (w \times u) = (w \times u) \cdot v = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$

And expanding this determinant along the bottom row also gives

$$v \cdot (w \times u) = v_1(w_2u_3 - w_3u_2) - v_2(w_1u_3 - w_3u_1) + v_3(w_1u_2 - w_2u_1).$$

Hence $u \cdot (v \times w) = v \cdot (w \times u)$. Now, substituting $v \to u$, $w \to v$, and $u \to w$ into $u \cdot (v \times w) = v \cdot (w \times u)$ yields $v \cdot (w \times u) = w \cdot (u \times v)$.

3. (10 pts) State and prove the Triangle Inequality for two vectors $u, v \in \mathbb{R}^n$. You may assume that the Cauchy-Schwarz Inequality is true.

The Triangle Inequality says that if $u, v \in \mathbb{R}^n$ then

$$||u + v|| \leq ||u|| + ||v||.$$

The square of the left-hand side is

$$||u + v||^2 = (u + v)(u + v) = ||u||^2 + 2uv + ||v||^2.$$

The square of the right-hand side is

$$\big(||u|| + ||v||\big)^2 = ||u||^2 + 2||u||\,||v|| + ||v||^2.$$

The Cauchy-Schwarz Inequality says that

$$|uv| \leq ||u||\,||v||.$$

Now, uv is a real number, so $uv \leq |uv| \leq ||u||\,||v||$. Hence

$$||u||^2 + 2uv + ||v||^2 \leq ||u||^2 + 2||u||\,||v|| + ||v||^2.$$

It follows that

$$||u + v||^2 \leq \big(||u|| + ||v||\big)^2$$

Since the square root function $g(x) = \sqrt{x}$ is an increasing function,

$$||u + v||^2 \leq \big(||u|| + ||v||\big)^2 \implies \sqrt{||u + v||^2} \leq \sqrt{\big(||u|| + ||v||\big)^2}.$$

Finally, note that $||u + v|| \geq 0$ and $||u|| + ||v|| \geq 0$, so

$$\sqrt{||u + v||^2} = ||u + v|| \qquad \text{and} \qquad \sqrt{\big(||u|| + ||v||\big)^2} = ||u|| + ||v||.$$

We can now conclude that

$$||u + v|| \leq ||u|| + ||v||.$$

4. (10 pts) Let $u, v, w \in \mathbb{R}^3$. Show that the triple product $(u \times v) \cdot w$ is equal to the determinant of the 3×3 matrix whose rows are u, v, and w in that order.

Let $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$, and $w = (w_1, w_2, w_3)$. Then

$$\begin{aligned}(u \times v) \cdot w &= \big(u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1\big) \cdot (w_1, w_2, w_3) \\ &= w_1(u_2v_3 - u_3v_2) - w_2(u_1v_3 - u_3v_1) + w_3(u_1v_2 - u_2v_1)\end{aligned}$$

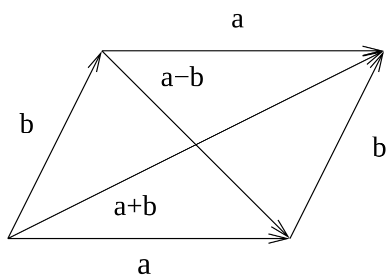
Now expanding the determinant along the bottom row, we get

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = w_1(u_2v_3 - u_3v_2) - w_2(u_1v_3 - u_3v_1) + w_3(u_1v_2 - u_2v_1)$$

which is the same as $(u \times v) \cdot w$.

5. (10 pts) **Extra credit problem.** When building a house, it is relatively easy to make sure that the two pairs of opposite sides in a room are equal. (If you are curious why, it is because you can build the frames for the walls in advance and lay the opposite ones on each other to make sure they are the same size before you nail them together at the corners.) A bigger challenge is to make sure that the corners are right angles. As you probably realize, a quadrilateral whose opposite sides are pairwise equal is definitely a parallelogram, but not necessarily a rectangle. An old trick, well-known to carpenters, to check if a room is "square" (which really means rectangle in carpenter lingo) is to measure the diagonals of the room, and if they are equal the room must be a rectangle. Now, your typical carpenter may not know how to prove that a parallelogram whose diagonals are equal must be a rectangle. But your knowledge of vectors puts you in a very good position to find such a proof. So, use vectors to prove that a parallelogram whose diagonals are equal must be a rectangle.

Let a and b be vectors along two adjacent sides of a parallelogram, as in the diagram below.



Note that the two diagonals are $a+b$ and $a-b$. Suppose these are of the same length. Then

$$\begin{aligned} ||a+b| &= ||a-b|| \\ ||a+b||^2 &= ||a-b||^2 \\ (a+b)(a+b) &= (a-b)(a-b) \\ a^2 + 2ab + b^2 &= a^2 - 2ab - b^2 \\ 2ab &= -2ab \\ 4ab &= 0 \\ ab &= 0 \end{aligned}$$

This shows that a and b must be perpendicular. Therefore all of the angles of the parallelogram are right angles, and the parallelogram is a rectangle.