MCS 222 Exam 2 Solutions

May 1, 2020

1. (5 pts each)
   (a) Describe the geometric meaning of the following mapping $(r, \theta, z) \mapsto (-r, \theta - \pi/4, z)$ in cylindrical coordinates. Be sure to justify your answer.

   The mapping $(r, \theta) \mapsto (-r, \theta)$ in polar coordinates would reverse the direction of a position vector in the xy-plane. That is it is a reflection across the origin. Or it can be viewed as a $180°$ rotation about the origin. By extension, the mapping $(r, \theta, z) \mapsto (-r, \theta, z)$ is a $180°$ rotation about the z-axis. Similarly, the mapping $(r, \theta, z) \mapsto (r, \theta - \pi/4, z)$ is a $45°$ rotation about the z-axis in the negative direction, that is clockwise. The composition of these two mappings is exactly $(r, \theta, z) \mapsto (-r, \theta - \pi/4, z)$, which must then be either a $180° - 45° = 135°$ rotation about the z-axis in the positive direction, that is counterclockwise, or a $180° + 45° = 225°$ rotation about the z-axis in the negative direction, that is clockwise.

   (b) Describe a hemisphere of diameter 5 units using inequalities. State the coordinate system used.

   This can be done in any coordinate system, but the easiest one to use is spherical coordinates. In those, $0 \leq \rho \leq 5/2$ describes a sphere of radius $5/2$ or diameter 5 centered at the origin. To make it a hemisphere, we can either restrict θ to half its usual range, that is $0 \leq \theta \leq \pi$, or we can restrict ϕ to half its usual range, that is $0 \leq \phi \leq \pi/2$. So both

$$0 \leq \rho \leq \frac{5}{2}, 0 \leq \theta \leq \pi, 0 \leq \phi \leq \pi$$

   and

$$0 \leq \rho \leq \frac{5}{2}, 0 \leq \theta < 2\pi, 0 \leq \phi \leq \frac{\pi}{2}$$

   describe hemispheres of diameter 5, although not the same ones. Note that the latter can also be written more concisely as

$$0 \leq \rho \leq \frac{5}{2}, 0 \leq \phi \leq \frac{\pi}{2}$$

   since any value of θ would be equivalent to some value between 0 and 2π anyway.
   In cylindrical coordinates, $r^2 + z^2 \leq (5/2)^2$ would describe a sphere of radius $5/2$ or diameter 5 centered at the origin. To make it a hemisphere, we can either restrict θ to half its usual range, that is

$$r^2 + z^2 \leq (5/2)^2, 0 \leq \theta \leq \pi$$

   or we could take the upper half of the sphere by restricting z to be nonnegative:

$$0 \leq r \leq \frac{5}{2}, 0 \leq \theta < 2\pi, 0 \leq z \leq \sqrt{\frac{25}{4} - r^2}.$$

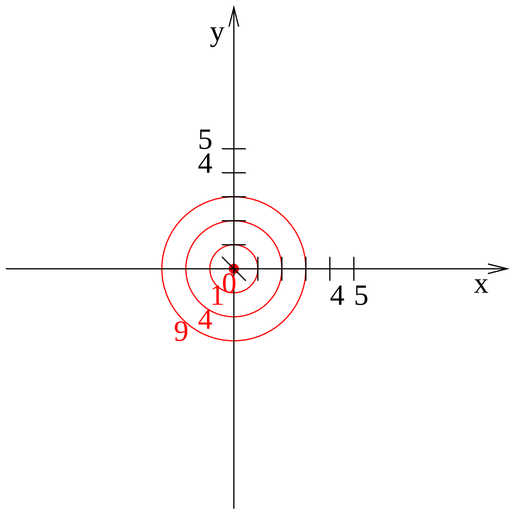
   In Cartesian coordinates, $x^2 + y^2 + z^2 \leq (5/2)^2$ would be a sphere of radius $5/2$ or diameter 5 centered at the origin. To take half of it, we could let

$$-\frac{5}{2} \leq x \leq \frac{5}{2}, -\sqrt{\frac{25}{4} - x^2} \leq y \leq \sqrt{\frac{25}{4} - x^2}, 0 \leq z \leq \sqrt{\frac{25}{4} - x^2 - y^2}$$

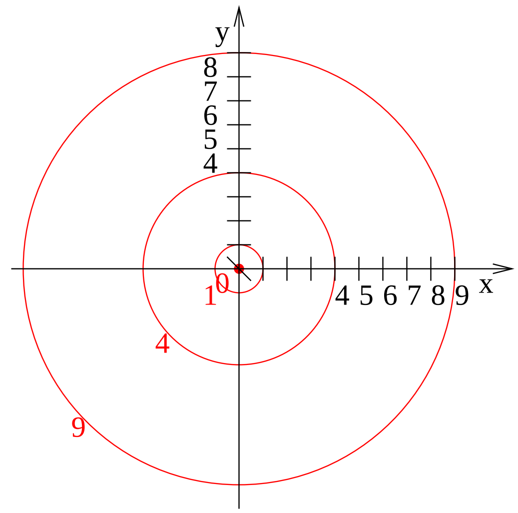
   to describe the half that is above the xy-plane.

2. (a) (7 pts) Sketch level sets of values $c = 0, 1, 4, 9$ for both $f(x, y) = x^2 + y^2$ and $g(x, y) = \sqrt{x^2 + y^2}$.

The level sets of both f and g are circles centered at the origin. The difference is that in the case of f, the levels $c = 0, 1, 4, 9$ correspond to circles $x^2 + y^2 = 0$, $x^2 + y^2 = 1$, $x^2 + y^2 = 4$, and $x^2 + y^2 = 9$, that is circles of radii $0, 1, 2, 3$, while in the case of g, these levels correspond to circles $x^2 + y^2 = 0$, $x^2 + y^2 = 1$, $x^2 + y^2 = 16$, and $x^2 + y^2 = 81$, that is circles of radii $c = 0, 1, 4, 9$. In other words, a circle of radius r centered at the origin is the level set of level r^2 for f, but the same circle is the level set of level r for g.



The level sets of f



The level sets of g

(b) (3 pts) How are the graphs of f and g different? (Hint: think carefully about what the spacing of the level curves you found in part (a) tells you about how steeply the surfaces rise?)

The level curves are circles centered at the origin in both cases. But in the case of f, the circle of radius r, $x^2 + y^2 = r^2$ corresponds to level $c = r^2$, whereas in the case of g, the same circle corresponds to level $c = r$. Another way to look at it is that in the case of f, the circles that correspond to the levels $c = 0, 1, 4, 9$ are of radii $0, 1, 2, 3$, whereas in the case of g, the circles that correspond to the levels $c = 0, 1, 4, 9$ are of radii $0, 1, 4, 9$. Either way we look at it, it is apparent that the graph of f increases with the square of the distance of (x, y) from the origin, whereas the graph of g increases linearly with that distance. So the graph of f is a paraboloid, while the graph of g is an upside down cone.

3. (10 pts) Let r be a positive real number and x_0 an element of $\mathbb{R}^n$. Prove that the open disk of radius r centered at x_0

$$D_r(x_0) = \{x \in \mathbb{R}^n \mid ||x - x_0|| < r\}$$

is an open subset of $\mathbb{R}^n$.

See Theorem 1 in Section 2.2.

4. (a) (4 pts) Let $f : \mathbb{R}^n \to \mathbb{R}$ and $x = (x_1, \ldots, x_n)$ an point in $\mathbb{R}^n$. State the definition of the partial derivative $\frac{\partial f}{\partial x_i}$.

$$\frac{\partial f}{\partial x_i} = \lim_{h\to 0} \frac{f(x_1,\ldots,x_{i-1},x_i+h,x_{i+1},\ldots,x_n) - f(x_1,\ldots,x_n)}{h}$$

(b) (6 pts) Recall that $\mathbb{R}^+$ is the set of positive real numbers. Let $S = \{(x,y,z) \mid x,y,z \in \mathbb{R}^+\} \subseteq \mathbb{R}^3$ and let $f : S \to \mathbb{R}$ be the function

$$f(x,y,z) = 2x^z - \sin\left(\frac{y}{z}\right).$$

Find all of the partial derivatives of f. You do not need to use the definition of the derivative to find the partial derivatives.

Keeping two of the variables constant and using the usual rules of single variable differentiation to differentiate with respect to the third, we get

$$\frac{\partial f}{\partial x} = 2zx^{z-1}$$

$$\frac{\partial f}{\partial y} = -\cos\left(\frac{y}{z}\right)\frac{1}{z}$$

$$\frac{\partial f}{\partial z} = 2x^z\ln(x) - \cos\left(\frac{y}{z}\right)y(-z^{-2}) = 2x^z\ln(x) + \frac{y}{z^2}\cos\left(\frac{y}{z}\right)$$

5. (10 pts) **Extra credit problem.** Let $f : \mathbb{R}^n \to R^{\geq 0}$ be the function $f(x) = ||x||$. That is f maps a vector x to its length. Show that f is continuous at every $x_0 \in \mathbb{R}^n$.

We need to show

$$\lim_{x\to x_0} f(x) = f(x_0),$$

which is

$$\lim_{x\to x_0} ||x|| = ||x_0||.$$

In terms of the $\delta - \epsilon$ definition of the limit, we want to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $||x - x_0|| < \delta$ then $\big|||x|| - ||x_0||\big| < \epsilon$. (Note that we do not need to make sure that $0 < ||x - x_0||$ in this case, because if $x = x_0$, then $||x|| - ||x_0|| = 0$ anyway.)

So let $\epsilon > 0$. First, note that

$$||x|| = ||x - x_0 + x_0|| \leq ||x - x_0|| + ||x_0||$$

by the Triangle Inequality. Hence

$$||x|| - ||x_0|| \leq ||x - x_0||.$$

Similarly,

$$||x_0|| = ||x_0 - x + x|| \leq ||x_0 - x|| + ||x||$$

and hence

$$||x_0|| - ||x|| \leq ||x_0 - x|| = ||x - x_0||.$$

Multiplying both sides by -1 yields

$$||x|| - ||x_0|| \geq -||x - x_0||.$$

Therefore

$$-||x - x_0|| \leq ||x|| - ||x_0|| \leq ||x - x_0||$$

and so

$$\big|||x|| - ||x_0||\big| \leq ||x - x_0||$$

This suggests that choosing $\delta = \epsilon > 0$ will do what we want. Indeed, if $||x - x_0|| < \delta = \epsilon$ then

$$\big|||x|| - ||x_0||\big| \leq ||x - x_0|| < \epsilon.$$

We can now conclude that

$$\lim_{x \to x_0} ||x|| = ||x_0||$$

and thus $f(x) = ||x||$ is continuous at any $x_0 \in \mathbb{R}^n$.