Notes for Section 1.4

We have already talked about cylindrical coordinates and a little about spherical coordinates too. Figure 1.4.5 is your key to understanding the relationship between the two. First, let P be the point whose Cartesian coordinates are (x, y, z). Let us find the spherical coordinates of P. Note that the first spherical coordinate ρ is just the distance from the origin to P, which is the same thing as the length of the vector $\vec{P} = (x, y, z)$. That is

$$\rho = |\vec{P}| = \sqrt{x^2 + y^2 + z^2}.$$

The second coordinate θ is the same as the θ in the polar coordinates of the point (x, y) in the xy-plane. We already know that one way to find θ is to use

$$\tan(\theta) = \frac{y}{x}.$$

This is the same way we found θ when converting to cylindrical coordinates. As we noted then, this gives two possible values for θ in $[0, 2\pi)$ because the tangent function has period π. To determine which of the two values is correct, you need to take into account which quadrant (x, y) is in. Also, if $x = 0$, the value of y/x is undefined. In this case, θ must be either $\pi/2$ or $3\pi/2$, and the correct value depends on the sign of y.

There is a way to avoid this somewhat annoying ambiguity with the tangent function. Look at figure 1.4.1 in the textbook, and notice that we also know

$$\sin(\theta) = \frac{y}{\sqrt{x^2 + y^2}} \qquad \text{and} \qquad \cos(\theta) = \frac{x}{\sqrt{x^2 + y^2}}.$$

In these, the denominator is 0 only if $(x, y) = (0, 0)$, in which case any value in $[0, 2\pi)$ will work for θ anyway. In general, neither of the equations above determines the value of θ uniquely because for any number in $(-1, 1)$, there are two possible angles in $[0, 2\pi)$ whose sine or cosine is exactly that number. For example, $\sin(\pi/6)$ and $\sin(5\pi/6)$ are both $1/2$ and $\cos(\pi/3)$ and $\cos(5\pi/3)$ are also both $1/2$. But there is only one angle in $[0, 2\pi]$ that satisfies both equations above at the same time. So instead of using the tangent to find θ along with considering which quadrant (x, y) is in, you could solve the sine and the cosine equations above and choose the only value that satisfies both. The same thing is true when you want to find cylindrical coordinates from Cartesian coordinates.

But back to spherical coordinates. The last ingredient we need is the value of ϕ. Look at figure 1.4.5 again. The right triangle whose hypotenuse is ρ and whose legs are r and z is the key. The angle between ρ and z is ϕ because the side z is parallel to the z-axis. So

$$\cos(\phi) = \frac{z}{\rho} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}.$$

You could get this same equation by noting that

$$\vec{k} \cdot \vec{P} = \|\vec{k}\| \, \|\vec{P}\| \cos(\phi)$$

and solving for $\cos(\phi)$. In any case, this determines ϕ uniquely because for any number in $[-1, 1]$, there is only angle in $[0, \pi]$ whose cosine is exactly that number. OK, there is one exception: if $\rho = 0$, then z/ρ is undefined. But this can only happen if $P = (0, 0, 0)$, in which case it does not matter what number we choose for ϕ.

In summary, the formulas you can use to convert from Cartesian to spherical coordinates are

$$\begin{aligned}
\rho &= \sqrt{x^2+y^2+z^2} \\
\tan(\theta) &= \frac{y}{x} \\
\sin(\theta) &= \frac{y}{\sqrt{x^2+y^2}} \\
\cos(\theta) &= \frac{x}{\sqrt{x^2+y^2}} \\
\cos(\phi) &= \frac{z}{\sqrt{x^2+y^2+z^2}}
\end{aligned}$$

along with the considerations about choosing the right value for θ I discussed above. Let us do an example.

Example 1. *Let P be $(2, 2, \sqrt{6})$ in Cartesian coordinates. Let us find the spherical coordinates of P.*

First,

$$\rho = \sqrt{2^2 + 2^2 + \sqrt{6}^2} = \sqrt{4+4+6} = \sqrt{14}.$$

Now,

$$\tan(\theta) = \frac{2}{2} = 1$$

so $\theta = \pi/4$ or $5\pi/4$. Of these, $\pi/4$ is in the first quadrant while $5\pi/4$ is in the third quadrant. Since $(2, 2)$ is in the first quadrant, the correct value is $\theta = \pi/4$. Finally,

$$\cos(\phi) = \frac{\sqrt{6}}{\sqrt{14}} = \frac{\sqrt{3}}{\sqrt{7}} = \sqrt{\frac{3}{7}},$$

so $\phi = \arccos(\sqrt{3/7}) \approx 0.857$.

Now, let us figure out how to convert from spherical coordinates to Cartesian coordinates. The story is again in Figure 1.4.5. Suppose that P is the point with spherical coordinates (ρ, θ, ϕ). We will find the Cartesian coordinates of P. First, using the right triangle in the diagram,

$$z = \rho\cos(\phi) \qquad \text{and} \qquad r = \rho\sin(\phi)$$

Now, we just do a conversion from polar coordinates (r, θ) to Cartesian coordinates (x, y):

$$\begin{aligned}
x &= r\cos(\theta) = \rho\sin(\phi)\cos(\theta) \\
x &= r\sin(\theta) = \rho\sin(\phi)\sin(\theta)
\end{aligned}$$

This is very straightforward and there is no ambiguity to deal with. Since it is just a matter of substituting into the above formulas, I will not do an example here. You can look at Example 2 in your textbook.

Example 3 is interesting, but not very well motivated at this point. Look at it, enjoy it, and do not stress if you do not quite understand the purpose of the example. Especially do not stress about the brief discussion about e_r, e_z, e_ρ, e_θ, and e_ϕ. These are called local coordinate systems, and we may encounter them later on, at which point the discussion will likely make more sense. If you wonder why the angles in figure 1.4.8 are q and f and the local coordinate vectors are e_q and e_f, it is because the Greek letters θ, ϕ, and ρ got unintentionally converted to the Roman alphabet during the typesetting of the book.