Notes for Section 2.1

Let me remind you of a few key concepts before you read this section. First, here is the informal definition of a function.

Let S and T be nonempty sets. A *function* (or *map* or *mapping*) f from S to T is a rule that assigns to every element $x \in S$ exactly one element $y \in T$. The set S is called the *domain* or *source* of f and the set T is the *codomain* or *target*. The usual notation for "f assigns y to x" is $f(x) = y$, where x is often referred to as the input or input value and y as the output or output value or simply value. The reason this is an informal definition is that there is no clearly defined meaning to what a rule is. It is left to your intuition that in everyday usage a rule implies consistence, i.e. for a particular input x the function must always assign the same output y.

In single variable calculus, you studied functions whose domain and codomain were subsets of the real numbers. Multivariable calculus studies functions whose domain and codomain are subsets of higher dimensional real vector spaces, namely $S \subseteq \mathbb{R}^n$ and $T \subseteq \mathbb{R}^m$. So we could write $f(\vec{x}) = \vec{y}$ where $\vec{x} \in \mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^m$, or

$$f(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_m).$$

It is important to understand that while we often visualize such vectors as arrows in a geometric space, they need not have such geometric meaning in a particular context. For example, the function $f(r, h) = \pi r^2 h$ gives the volume of a circular cylinder of radius r and height h, and $(r, h) \in \mathbb{R}^2$, but it does not make much sense to think about r, h as a location in the plane in the context of this example. In fact, we often think of the input of f not as one variable, whose value is a vector in n-dimensional space, but as n variables, each of which is a real number. Here is another example, which may help you understand what I am talking about. According to physics, a fixed amount of gas, say 100 g of CO_2 in a closed container exerts a pressure on the sides of that container that is determined by the volume of the container and the temperature of the gas. (It is actually not the mass of the gas that matters but the number of molecules, but 100 g of CO_2 would always have the same number of molecules independent of volume and temperature.) So there is some function $g(V, T) = p$, where V and T are the volume and the temperature and p is the pressure. While $(V, T) \in \mathbb{R}^2$, it makes more practical sense to think of V and T as two separate variables rather than as coordinates of one vector.

In prior math classes, you often studied the graph of a function $f : S \to T$ whose domain and codomain were subsets of $\mathbb{R}$. Such a graph consists of points of the form $(x, f(x))$ in $\mathbb{R}^2$. In fact, the graph of f is the set

$$G(f) = \{(x, f(x)) \mid x \in S\} \subseteq \mathbb{R}^2.$$

Actually, the following definition works for any function $f : S \to T$ can be defined this way, no matter what sets S and T are.

Definition 1. Let $f : S \to T$ be a function. The *graph* of f is the subset

$$G(f) = \{(x, f(x)) \mid x \in S\}$$

of the Cartesian product $S \times T$.

What is the Cartesian product of two sets?

Definition 2. The *Cartesian product* of two sets S and T is the set

$$S \times T = \{(s, t) \mid s \in S, t \in T\}.$$

So the Cartesian product $S \times T$ consists of ordered pairs such that the first element of the pair is an element of S and the second element is an element of T.

Example 1. *If* $S = \{cat, mouse\}$ *and* $T = \{1, 2\}$ *then*

$$S \times T = \{(cat, 1), (cat, 2), (mouse, 1), (mouse, 2)\}.$$

Another example of the Cartesian product, which you are quite familiar with, is

$$\underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}} = \mathbb{R}^n.$$

If $f : S \to T$ where $S, T \subseteq \mathbb{R}$, then the graph of f can be visualized as a curve in the plane $\mathbb{R}^2$ where the first coordinate is used for the input and the second for the output of f. You are familiar with such graphs. In multivariable calculus, the graph of a function $f : S \to T$, where $S \subseteq \mathbb{R}^n$ and $T \subseteq \mathbb{R}^m$ would consist of points of the form $(\vec{x}, f(\vec{x}))$ or $\big((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_m)\big)$ where $(y_1, y_2, \ldots, y_m) = f(x_1, x_2, \ldots, x_n)$ in $S \times T$, which in turn is a subset of $\mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$. In fact, we would normally write $\big((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_m)\big)$ simply as $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$. While such a graph is a perfectly legitimate mathematical object and can be studied and understood using the tools of mathematics, visualizing it is difficult unless $n+m \leq 3$. That is quite a restriction. But visualization is important for developing conceptual understanding, and often enough, what we learn by studying examples in small dimensions readily generalizes to higher dimensions. So much of Section 2.1 is about the various ways that the graph of a multivariable function can be visualized. If you think about it, we are typically even more limited than the inequality $n + m \leq 3$ would suggest because our standard medium of communication is the 2-dimensional plane of a sheet of paper or computer screen. But our brains understand 3-dimensional space and over time people have developed a number of ways to represent 3-dimensional objects in 2 dimensions in such a way that our brains can readily reinterpret them as 3-dimensional.

For now, we will restrict our attention to functions $f : S \to T$ where $S \subseteq \mathbb{R}^n$ and $T \subseteq \mathbb{R}$. Such functions are called *real-valued*. If $n = 2$, the graph of such a function is a subset of $\mathbb{R}^3$. But it cannot just be any subset, it is called a surface. Roughly speaking, a surface is what you get by deforming a plane, allowing it to curve, stretch, fold, etc. It is customary to refer to use x, y, and z to refer to the three coordinates in $\mathbb{R}^3$. By convention, when the domain of f is in $\mathbb{R}^2$ and its codomain is in $\mathbb{R}$, we use x and y as the input variables and z as the output.

At this point, I recommend that you play a little with the surface plotter app in Desmos, which is linked from the class website. Make up some formulas for functions $f(x, y)$ and look at their graphs. For now, don't worry about how those graphs are made, your brain will know how to make sense of them as surfaces in 3-dimensional space.

One way to visualize such a surface $z = f(x, y)$ is to look at its *contours* or *level curves*, which you get by setting the output equal to a fixed value c, as in $f(x, y) = c$. So the level curve of f at level c is the set

$$\{(x, y) \mid f(x, y) = c\} \subseteq \mathbb{R}^2$$

and typically looks like a curve in the xy-plane. You can visualize the surface $z = f(x, y)$ in 2 dimensions by plotting some of its level curves in the xy-plane. Chances are you are familiar with such plots because this is what topographic maps use to represent elevation above sea level or depth of lakes, seas, and oceans below sea level (technically, those or called bathymetric maps). Contour plots are also commonly used in meteorological maps of temperature, pressure etc. Another way you can think about contours or level curves is that these are what you get if you slice the graph of f with planes parallel to the xy-plane. Study Examples 1-4 which show you graphs of various $\mathbb{R}^2 \to \mathbb{R}$ functions and the corresponding contour plots. You can make your own contour plots. There are a number of tools on the internet that do this. One is Wolfram Alpha (this link is also on our class website). Entering

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plot3d(f(x,y)=x^2-y^2)
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in Wolfram Alpha will give you both the 3-D plot and the contour plot. Play with different functions. There are many other sites on the internet for this. Another that is linked from your class website is math3d.org. Look around to see if you can find more that you really like.

Another way to visualize surfaces in 3 dimensions is to slice them with planes that are perpendicular to the xy-plane. These are called sections. In particular, you could use planes that are parallel to the xz-plane and the yz-plane. If you think about it a bit, this means that you are setting either x equal to some fixed value and letting only y vary, or vice versa. This is exactly what the plotting app in Desmos does. The result probably also looks familiar to you because it is a commonly used method in computer graphics.

Both of these ideas can be generalized to real-valued functions of n input variables. You can defined the *level sets* of such a function as

$$\{(x_1, \ldots, x_n) \mid f(x_1, \ldots, x_n) = c\} \subseteq \mathbb{R}^n.$$

You can also study surfaces in higher dimensions by looking at their sections by lower-dimensional spaces. Both of these are more theoretical tools than practical visualization tools, as you'd have to be like Cypher in the Matrix to be able to visualize the original surface based on its sections.