Notes for Section 2.4

This section is about paths and curves in $\mathbb{R}^n$ and is not as loaded with abstraction as the last two sections. A *path* is a special kind of multivariable function $f : I \to R^n$ where I is an interval in $\mathbb{R}$. The interval I is often a closed interval, but it does not have to be. The book's favorite notation is c for a path and t for the input variable of c. Look at the simple Examples 1-3 to understand why we would refer to such a function as a path and familiarize yourself with the terminology. The difference between a *curve* and a path may seem mysterious at first. A curve in $\mathbb{R}^n$ is the image of a path $c : I \to \mathbb{R}^n$. The path is a function and the curve is a subset of $\mathbb{R}^n$. A particular curve could be the image of several different paths. For example, the circle

$$C = \{(x, y) \mid x^2 + y^2 = 1\}$$

in $\mathbb{R}^2$ is the image of both $c_1, c_2 : [0, 2\pi] \to \mathbb{R}^2$ given by $c_1(t) = (\cos(t), \sin(t))$ and $c_2(t) = (\cos(t), -\sin(t))$, but these two paths are not the same as $c_1(t) \neq c_2(t)$ for most values of t.

Since a path is a vector-valued function, working with paths is very much a vector game. Example 4 is a more elaborate example of a path and illustrates how you can use vector algebra to construct a path $c(t)$ that describes the position of a moving object in $\mathbb{R}^n$ as a function of time. This is a rather typical application of paths, but it is not the only thing that paths are good for. Inspired by this example, we can interpret the derivative $\frac{dc}{dt}$ or $c'(t)$ as the velocity of c using the usual definition

$$\frac{dc}{dt} = \lim_{h \to 0} \frac{c(t+h) - c(t)}{h}.$$

Note that this is not a partial derivative as c has only one input variable. This is why the c' notation also makes sense: it does not lead to any doubt with respect to which variable we are differentiating. The numerator of the difference quotient is a vector, but the denominator is a number, and dividing a vector by the number h makes sense because we can always view it as multiplying by $1/h$. In fact, if you view $c(t)$ in terms of component functions

$$c(t) = \big(c_1(t), c_2(t), \ldots, c_n(t)\big)$$

then

$$\begin{aligned}
c'(t) &= \lim_{h \to 0} \frac{\big(c_1(t+h), \ldots, c_n(t+h)\big) - \big(c_1(t), \ldots, c_n(t)\big)}{h} \\
&= \lim_{h \to 0} \left(\frac{c_1(t+h) - c_1(t)}{h}, \ldots, \frac{c_n(t+h) - c_n(t)}{h} \right) \\
&= \left(\lim_{h \to 0} \frac{c_1(t+h) - c_1(t)}{h}, \ldots, \lim_{h \to 0} \frac{c_n(t+h) - c_n(t)}{h} \right) \\
&= \big(c_1'(t), \ldots, c_n'(t)\big),
\end{aligned}$$

where the reason we can distribute the limit over the component functions is Theorem 3.(v) in Section 2.2. So this is really good old single variable calculus. It follows that each component function c_i is differentiable at t_0 if and only if its derivative exists there. And if each component is differentiable, then so is c. This is not completely obvious, but if you think about what differentiability means for c, you may be able to convince yourself that if each component function c_i has a good linear approximation l_i then the linear function $l : \mathbb{R} \to \mathbb{R}^n$ given by

$$l(t) = \big(l_1(t), \ldots, l_n(t)\big)$$

is a good approximation to c in the usual sense.

Note that the velocity c' of c is another vector-valued function from I to $\mathbb{R}^n$. If you want to talk about the speed at which the object whose motion c describes moves, you need to take the magnitude $||c'||$.

You can use the velocity $c'(t_0)$ to write down the equation of a tangent line to c at the point t_0 using the usual formula

$$l(t) = c'(t_0)(t - t_0) + c(t_0).$$

This is because the vector $c'(t_0)$ itself is also tangent to the path c at t_0. That this is so is relatively easy to see if you visualize what the secant vector

$$\frac{c(t_0 + h) - c(t_0)}{h}$$

looks like and what happens to it as $h \to 0$.

Alright, the rest is a matter of practice at using these concepts. Look at the remaining Examples 5-9, try to understand them, and let me know if you would like to talk about any one of them in class.