MCS 314 Exam 1 Solutions

1. Let $n \in \mathbb{Z}^{\geq 3}$. For a permutation $\sigma \in S_{n-2}$, define $\sigma' \in S_n$ by
$$\sigma'(i) = \begin{cases} \sigma(i) & \text{if } 1 \leq i \leq n-2 \\ i & \text{if } n-1 \leq i \leq n. \end{cases}$$
Now, let $\theta : S_{n-2} \to A_n$ be the map
$$\theta(\sigma) = \begin{cases} \sigma' & \text{if } \sigma \in A_{n-2} \\ \sigma'(n-1\ n) & \text{if } \sigma \notin A_{n-2}. \end{cases}$$

(a) (3 pts) Verify that θ indeed maps every element of S_{n-2} to some element in A_n.

Hint: You may want to start by showing that $\phi : S_{n-2} \to S_n$ given by $\phi(\sigma) = \sigma'$ is a homomorphism that maps transpositions in S_{n-2} to transpositions in S_n.

Let $\sigma, \tau \in S_{n-2}$ and let $\Omega = \{1, 2, \dots n-2\}$. If $i \in \Omega$, then $\tau(i) \in \Omega$, and hence
$$\phi(\sigma\tau)(i) = (\sigma\tau)'(i) = \sigma\tau(i) = \sigma(\tau(i)),$$
and
$$\phi(\sigma)\phi(\tau)(i) = \sigma'\tau'(i) = \sigma'(\tau'(i)) = \sigma'(\underbrace{\tau(i)}_{\in\Omega}) = \sigma(\tau(i)).$$
If $i = n-1$ or $i = n$, then
$$\phi(\sigma\tau)(i) = (\sigma\tau)'(i) = i,$$
and
$$\phi(\sigma)\phi(\tau)(i) = \sigma'\tau'(i) = \sigma'(\tau'(i)) = \sigma'(i) = i.$$
Hence $\phi(\sigma\tau) = \phi(\sigma)\phi(\tau)$. This shows that ϕ is a homomorphism.
It should be clear that if $\sigma = (ij)$ in S_{n-2}, then $\phi(\sigma) = \sigma' = (ij)$ in S_n. So ϕ maps transpositions to transpositions. It now easily follows that if σ is a product of k transpositions in S_{n-2} then $\phi(\sigma)$ is also a product of k corresponding transpositions in S_n. Hence ϕ maps even permutations to even permutations and odd permutations to odd permutations. Therefore if σ is even, then $\theta(\sigma) = \sigma'$ is also even; and if σ is odd, then $\theta(\sigma) = \sigma'(n-1\ n)$ is again even. So $\theta(\sigma) \in A_n$ for all $\sigma \in S_{n-2}$.

(b) (5 pts) Show that θ is a homomorphism whose kernel is $\{()\}$.

We have shown in part (a) that $(\sigma\tau)' = \sigma'\tau'$ for all $\sigma, \tau \in S_{n-2}$. Observe that if $\sigma \in S_{n-2}$, then σ' and $(n-1\ n)$ are disjoint permutations in S_n, and therefore they commute. Now, if $\sigma, \tau \in A_{n-2}$, then $\sigma\tau \in A_{n-2}$, and so
$$\theta(\sigma\tau) = (\sigma\tau)' = \sigma'\tau' = \theta(\sigma)\theta(\tau)$$
If $\sigma, \tau \notin A_{n-2}$, then σ and τ are odd permutations, and so $\sigma\tau$ is even. Hence
$$\theta(\sigma\tau) = (\sigma\tau)' = \sigma'\tau' = \sigma'\tau'(n-1\ n)^2 = \sigma'(n-1\ n)\tau'(n-1\ n) = \theta(\sigma)\theta(\tau).$$
If $\sigma \in A_{n-2}$ and $\tau \notin A_{n-2}$, then σ is even and τ is odd, and so $\sigma\tau$ is odd. Hence
$$\theta(\sigma\tau) = (\sigma\tau)'(n-1\ n) = \sigma'\tau'(n-1\ n) = \theta(\sigma)\theta(\tau).$$
If $\sigma \notin A_{n-2}$ and $\tau \in A_{n-2}$, then σ is odd and τ is even, and so $\sigma\tau$ is odd. Hence
$$\theta(\sigma\tau) = (\sigma\tau)'(n-1\ n) = \sigma'\tau'(n-1\ n) = \sigma'(n-1\ n)\tau' = \theta(\sigma)\theta(\tau).$$
In all four cases, $\theta(\sigma\tau) = \theta(\sigma)\theta(\tau)$. So θ is a homomorphism.

Finally, suppose $\theta(\sigma) = ()$. Then either $\sigma' = ()$ or $\sigma'(n-1\ n) = ()$. But the latter is not possible since $\sigma'(n-1\ n)$ maps $n-1$ to n, and so it cannot be the identity permutation. Therefore $\sigma' = ()$, which means $i = \sigma'(i) = \sigma(i)$ for all $1 \le i \le n-2$. This shows $\sigma = ()$. Therefore $\ker(\theta) = \{()\}$.

(c) (2 pts) Conclude that A_n contains a subgroup that is isomorphic to S_{n-2}.

Since $\ker(\theta) = \{()\}$, the map θ is injective. Therefore its image $\theta(S_{n-2})$, which is a subgroup of A_n is isomorphic to S_{n-2}.

2. Let G be a nonabelian group of order 6.

(a) (6 pts) Prove that G has a nonnormal subgroup H of order 2.

By Cauchy's Theorem, G must have an element x of order 2 and an element y of order 3. Let $H = \langle x \rangle$. Suppose H is normal in G. Let y be any element of G that is not in H. Since H is normal, its left and right cosets are the same. In particular,

$$yH = Hy \implies \{y, yx\} = \{y, xy\} \implies yx = xy.$$

Hence x and y commute. It follows that $K = \langle x, y \rangle$ is an abelian subgroup of G. Since $|K|$ contains x and y, the order of K must be divisible by both 2 and 3. The only possibility is that $|K| = 6$, and so $K = G$. But this results in a contradiction since G is nonabelian. Therefore H must not be normal in G.

(b) (4 pts) Use the result in part (a) to prove that G must be isomorphic to S_3.

Hint: Show that the action of G by left multiplication on the left cosets of H induces a permutation representation of G into S_3 whose kernel is trivial.

Let $A = \{H_1, H_2, H_3\}$ be the set of left cosets of H, such that $H_1 = H$. Let G act on A by left multiplication. We showed in class that this is a group action, and that the induced permutation representation $\phi : G \to S_A$ is a group homomorphism. Since $|A| = 3$, we may treat S_A as S_3 and π as a homomorphism $G \to S_3$. Now, suppose $g \in \ker(\pi)$. Then $gH_i = H_i$ for all $i = 1, 2, 3$. In particular, $gH = gH_1 = H_1 = H$. Hence $g \in G$. So $\ker(\pi) \le H$. But $\ker(\pi)$ is a normal subgroup of G, and H is not, so $\ker(\pi) \ne H$. Therefore $\ker(\pi) = \{1\}$. This shows that π is injective. Since $|G| = 6 = |S_3|$, the image of ϕ must be all of S_3. Hence $\phi : G \to S_3$ is an isomorphism.

3. (a) (4 pts) Let G be a group. State the Class Equation for G and explain what it means. You do not need to prove that the class equation holds, you just need to explain what the various parts of the equation mean.

The Class Equation is

$$|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|$$

where $g_1, g_2, \ldots, g_r$ are representatives of the nontrivial (that is of size more than 1) conjugacy classes of G, one representative from each such conjugacy class, and $Z(G)$ is the center of G.

(b) (6 pts) Let G be a group of prime power order $|G| = p^\alpha$ with $\alpha \in \mathbb{Z}^+$. Use the Class Equation to prove that the center of G must be nontrivial, that is $Z(G) \ne \{1\}$.

In the class equation for G, the term $|G : C_G(g_i)|$ is the size of the orbit of g_i under conjugation by the elements of G. The g_i are the representatives of the conjugacy classes of size greater than 1. So $|G : C_G(g_i)| > 1$. This number is also the index of the subgroup $C_G(g_i)$, so it must divide $|G| = p^\alpha$. Therefore $|G : C_G(g_i)|$ is some power p^{β_i} where $\beta_i \geq 1$. Hence the number on the right-hand side of

$$|Z(G)| = |G| - \sum_{i=1}^{r} |G : C_G(g_i)|$$

is divisible by p. Therefore $|Z(G)|$ is also divisible by p. This shows that $Z(G)| \neq 1$, that is the center of G is nontrivial.

4. (5 pts each) Let G be a group. For $g \in G$, let $\sigma_g : G \to G$ be the map

$$\sigma_g(x) = gxg^{-1}.$$

We proved in class that σ_g is an automorphism of G. Define the map $\phi : G \to \mathrm{Aut}(G)$ by $\phi(g) = \sigma_g$.

(a) Prove that ϕ is a homomorphism.

Let $g, h \in G$. Then $\phi(gh) = \sigma_{gh}$ and $\phi(g)\phi(h) = \sigma_g\sigma_h$. Now, let $x \in G$. Then

$$\sigma_{gh}(x) = (gh)x(gh)^{-1} = ghxh^{-1}g^{-1} = \sigma_g(hxh^{-1}) = \sigma_g(\sigma_h(x)) = \sigma_g\sigma_h(x).$$

Since this is true for all $x \in G$, we can conclude $\sigma_{gh} = \sigma_g\sigma_h$ and hence $\phi(gh) = \phi(g)\phi(h)$. Therefore ϕ is a homomorphism.

(b) Prove that $\ker(\phi) = Z(G)$, and use this to conclude that the image $\phi(G)$ is isomorphic to $G/Z(G)$.

Let $g \in G$. Then

$$\begin{aligned} g \in \ker(\phi) &\iff \sigma_g(x) = x \text{ for all } x \in G \\ &\iff gxg^{-1} = x \text{ for all } x \in G \\ &\iff g \in Z(G) \end{aligned}$$

So $\ker(\phi) = Z(G)$. But the First Isomorphism Theorem, $\phi(G) \cong G/Z(G)$.

5. (10 pts) **Extra credit problem.** Let G be a finite group of odd order. Show that if G has a normal subgroup N of order 5, then $N \leq Z(G)$.

Hint: Consider the conjugacy classes of the nonidentity elements in N and show that at least one of them is of size 1. Show that this implies that all of the nonidentity elements of N are in the center of G.

Suppose N is a normal subgroup of order 5 in G. Let x be any nonidentity element of N. Then $|x| = 5$ and $N = \langle x \rangle$. For every $g \in G$, the conjugate gxg^{-1} must be in N, because N is normal, and must also be order 5, because conjugation preserves the order. Hence conjugation by G permutes the nonidentity elements of N. Therefore, the size of the conjugacy class of x is $|G : C_G(x)| \leq 4$. This number must also divide $|G|$, which is odd, and therefore it must be either 1 or 3. This shows that the four nonidentity elements of N either form a conjugacy class of size 1 and a conjugacy class of size 3, or four conjugacy classes of size 1. In either case, there is a nonidentity element y of N that is in a conjugacy class by itself, that is $y \in Z(G)$. But y also generates N, so every other element of N must also be in $Z(G)$. Hence $N \leq Z(G)$.