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**Working Title of Article**: *Synthesizing Strategies Creatively: Solving Linear Equations*

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**Abstract:** (Give a 1- or 2-sentence overview of the article.)

With a lesson on solving equations, we discuss how effective lessons can result from (a) synthesizing strategies found in NCTM articles that tap the creativity of teachers, (b) sequencing these strategies from concrete to abstract, and (c) identifying in an explicit manner the core mathematical ideas embedded within the lesson.

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Linear Equations; Professional Development; Common Core Lesson Design;

In this article, we present an interactive approach to teach students how to solve linear equations. The approach combines strategies from several articles in Mathematics Teacher and Mathematics Teaching in the Middle School. We developed and tested this approach in practice by way of our professional development work with pre-service and in-service teachers. We also closely observed how one of the teachers refined this strategy to teach his students and assessed the students' skills to solve linear equations. Throughout the article we discuss the core mathematical ideas behind this approach and point out connections to the Common Core Mathematics Standards and Practices. Finally, we share this work hoping it can be helpful to teachers and coaches interested in finding new ways to teach traditional mathematics topics that are well aligned with the Common Core. Our intent is to tap into the natural desire of teachers to learn new strategies that can ignite their imagination to create new lessons or to adapt/replicate lessons created by others. To this end, we first present the details of a lesson for solving linear and literal equations as experienced by an algebra teacher and his students. We then explain how the use of ideas found in past NCTM journals helped bring this lesson to life.

## Tracing and Retracing the Moves

Walk into Mr. Arellano's class on the day he begins the unit on solving linear equations and you may wonder why students seem to be learning line dancing. "3 steps to the left; 5 steps back; 1 step to the right; 2 steps forward," Mr. Arellano tells his students as they try to figure him out. "Now take it back to the start, tracing back your steps from where you are!" Chaos ensues as most students struggle to know what to do. After a few moments he has the class start over but this time he asks them to listen carefully so they remember how to get back to the start. "3 steps to the left; 5 steps back; 1 step to the right; 2 steps forward; Now take it back to the start," he repeats. Few students figure out how to get back correctly, the majority of the students partially get back to the start, stopping somewhere in between; others just freeze trying to figure out what to do.

Kiddingly, Mr. Arellano tells the class he has seen them do *The Electric Slide* (Silver, 1976) at school dances with no problems, everyone in sync, following the steps to the music without hesitation. He challenges them to do the same with this new dance routine. Their eyes light up, ready to take on his challenge saying "Again Mr. Arellano!" Before starting, he asks students to share some ideas on how to get back to the start. One student shares, "you need to go backwards." Mr. Arellano asks for someone else to share what the student means by backwards. A student says that it means to go in the opposite direction, to which someone else says, "right goes left; forward goes backward." When asked if this is what she meant by backwards, the student says she meant something different, but agrees that one has to go in opposite directions to get back to the start. She then says that she meant the last step becomes the first step, and that the first step becomes the last step, in order to get back to the start. Someone else shares, "Ohhh. You mean in reverse order!!" "Important points to remember," Mr. Arellano says, "reverse order and opposite direction," as he gives the steps to the dance routine again. After a few more tries the class is in sync and heads back to the classroom.

Mr. Arellano continues by sharing how he gets from the gym to the school each morning. With a map compass on the board, he sketches the path as he gives the directions, "4 blocks north, 2 blocks east, 1 block south, 3 blocks west," asking the class to draw the sketch on their whiteboards as well. He then wonders aloud how to get back to the gym after school if he took the same path. While some students have a confident smile and are eager to give their answers, he notices other students are struggling to figure out what to do. Before opening the discussion, Mr. Arellano goes to the front of the class, and models the path, 4 steps north (back), 2 steps east (left), 1 step south (forward), 3 steps west (right). He proceeds by having students write down the directions he needs to follow to get back to the gym from school. Introducing Gratzer's (2003) *direction table* (explained in figure 1), he first completes the left-hand side of the table with the directions to get from the gym to the school. Only after there is consensus from the class is the right-hand side of the table completed as students determine together how to get back to the gym from the school (figure 2).

Mr. Arellano asks each student to draw the path they take from *their* home to school and to also write the directions on the left-hand side of a direction table. To complete the right-hand side of the table, he has students trace their path from school back to their home with their fingers, making sure they write down the directions on the table at the same time. Mr. Arellano gives a few more direction tables to complete independently before discussing them as a whole class. By now, students begin to see two kinds of changes in this process, (a) the order of the steps is reversed, and (b) the use of the opposite direction for each step. With these two insights firmly in place, Mr. Arellano explains how a *coding scheme* (Clausen 2005, summarized in figure 1) can be combined with a *direction table* into one tool (Gratzer 2003, figure 3) that will help them solve linear equations.

He then connects the previous two activities to the process of solving linear equations as suggested by Gratzer (2003) and Clausen (2005). Emphasizing that single-variable equations can be understood as questions whose answer are the values that make the equations true (Common Core State Standard 6EE-5), he clarifies that in a direction table the variable is now the starting point, the operations replace the directions, the number of steps or blocks are replaced by the numbers being used with each respective operation, and the ending point is the last equation with the solution, if any. Using problems similar to those provided by Clausen, e.g. figure 4, the class creates together the direction table for the equations and decides if the solution satisfies the original equation. Using the animation and timing features of PowerPoint ( see a brief YouTube video at **http://tinyurl.com/2013arellano )** Mr. Arellano helps the class shift from the use of directions tables to the use of algebra to solve linear equations. In the final part of the lesson, students begin to learn how to use a direction table to solve literal equations.

After so much doing and undoing (Driscoll, 1999, p. 15) of equations, an assessment at the end of the week showed students transitioning to solving equations such as $\frac{1}{6}x - \frac{5}{12} = \frac{1}{12}$ algebraically, with 41 of the 62 finding the correct solution (figure 5). But when asked to solve $3x + 7y = b$ for y, only 7 of the 62 students solved the problem correctly. It became clear to Mr. Arellano that shifting student thinking from solving single variable equations to solving literal equations was not a trivial transition for the class.

In the days that follow, Mr. Arellano keeps providing different types of linear and literal equations for students to solve. To facilitate the transition from the use of direction tables to the use of algebra, most problems are debriefed as a whole class using the *pass-the-pen* strategy (see Hawes 2007, summarized in figure 1). In this case, an equation is put on the board and Mr. Arellano gives a pen to one student to show the first step in solving it. After showing what to do, the student explains to the class the thinking behind what was done. The class then asks questions, proposes modifications, or accepts the work. Once there is consensus, the student hands the pen over to another student of her choosing to carry out the next step. Typically, the conversations revolve around the use of the two strategies and how they are related. In these discussions Mr. Arellano introduces key academic language, e.g. equivalent equations, explaining the meaning of these key mathematical ideas, e.g. equivalent equations have the same solution, and then has students use the academic language in subsequent explanations.

In a second assessment with two similar problems, 46 of the 62 students solved *both* problems correctly. Students were able to overcome the solving of literal equations through the use of a direction table as illustrated in figure 6. Encouraged by their improvement, Mr. Arellano continued using these ideas to teach the rest of the unit to his students. He also decided to start exploring NCTM’s journals again to find ideas to synthesize a new lesson about functions. Within the context of Mr. Arellano’s experience, we now provide a short description of the process and the three key principles that guided our work in putting this lesson together on solving linear equations.

## Opening More Doors to More Students

As the demand for ideas to implement the Common Core State Standards and the Standards for Mathematical Practices surges across the United States, teachers find themselves scrambling to find resources that will help them meet these standards. While some teachers are eager to come up with ideas from scratch, and others are just as eager to find ideas that are ready to implement as is, our approach was to find strategies to teach linear equations within NCTM’s journals to spark the creativity of teachers. At the time, we found articles from Borlaug (1997), Clausen (2005), Gratzer (2003), Hawes (2007), Hedin (2007), King (2003), Lee (2000), and Reeves and Gleichowski (2007). After reviewing these articles, we identified strategies from Clausen, Gratzer, and Hawes that would support building a lesson sequence to solve equations from the kinesthetic to the visual to the symbolic; that is from the ***concrete to the abstract***.

Mr. Arellano's lesson combined and adapted ideas from Gratzer (2003), Clausen (2005), and Hawes (2007) to form an introduction to the solving of linear and literal equations that is referenced throughout the rest of the unit, tying in together the Common Core Standards listed in figure 7. In effect, Gratzer's use of maps and direction tables, Clausen's use of a coding scheme to code operations, and Hawes pass-the-pen strategy to involve students became the ***pedagogical seeds*** that sparked Mr. Arellano's imagination. Specifically, he added the idea of line dancing so students would use their bodies to feel what happens when one retraces one's steps (kinesthetic), adapted the idea of using a map by having students make a map from their home to school (visual) because it would be more relevant to what they do each morning, merged the direction table and coding scheme into one integrated tool (visual and symbolic), used the timing features of PowerPoint to dynamically show how the coding scheme is connected to solving an equation algebraically (symbolic), and incorporated academic language (linguistic) into class discussions when passing the pen between students.

The lesson, as shared via Mr. Arellano and his students' experience, also incorporated the Standards of Mathematical Practice listed in figure 8. Students developed their skill to (a) reason abstractly and quantitatively as they learned how to contextualize (dance routine, direction table) and decontextualize (symbolic manipulation) the solving of linear equations and (b) look for and express regularity in repeated reasoning as the class realized through its discussions (pass-the-pen) which operations to use, and what order, to find the solution, if any, of an equation.

Essential to any math lesson is conveying a consistent message of the ***core mathematical ideas*** behind the concepts, procedures, and activities to form a strong foundation in students. Some of these ideas become more obvious when comparing strategies from different sources. For example, when comparing Gratzer (2003) and Clausen (2005), teachers readily recognize, as do students, the importance of "opposite directions," i.e. inverse operations, and "reverse order," i.e. undoing operations, to solve equations. Embedded within this process, however, is the bigger idea of equivalence (Charles, 2005) reflected by the resultant equivalent equations to find a solution to the equation. This idea then needs to be part of the conversation with and among students as they solve equations. At an even deeper level are the multiple roles of a variable in equations, e.g. *x* simply taking the place of a number that makes the equality $\frac{1}{6}x - \frac{5}{12} = \frac{1}{12}$ true vs. *x* and *b* serving as constant parameters in $3x + 7y = b$ while one solves for y, vs. covarying with another variable or set of variables, such as *x* and *y* in the set of ordered pairs that form the line $3x + 7y = 6$ (NCTM, 2000, p. 225). Learning to solve linear and literal equations at the same time helps make these multiple roles explicit for students.

We end this section by noting how the ideas in this lesson can be extended for use in future algebra classes by recognizing that the same notions of inverse operations carried out in reverse order apply to inverse functions. In fact, an analogous dance move/direction table lesson could be used to introduce inverse functions. Mathematics teachers may also recognize that the entries in a direction table are invertible functions from the real numbers to the real numbers, e.g. $M_2$ and $A_1$ are the $\mathbf{R} \to \mathbf{R}$ functions $M_2(x) = 2x$ and $A_1(x) = x + 1$ respectively, and listing their inverses in reverse order is just the standard algorithm for finding the inverse of an element in a group; in this case the group of invertible functions from the real numbers to the real numbers.

## Conclusion

Having observed other teachers having similar results to Mr. Arellano's, we are confident that the ideas shared in this article, i.e. the lesson itself and the manner in which the lesson came about, can likewise be useful to teachers and coaches. Rather than reinventing the wheel so to speak, we have demonstrated how effective lessons incorporating Common Core Standards and Practices can be designed by (a) synthesizing strategies found in NCTM articles that tap the creativity of teachers, (b) sequencing the strategies from the concrete to the abstract, and (c) identifying in an explicit manner the core mathematical ideas embedded within the lesson.

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