ON BRAIDED TENSOR CATEGORIES OF TYPE BCD

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ABSTRACT. We give a full classification of all braided semisimple tensor categories whose Grothendieck semiring is the one of $\mathrm{Rep}\,\big(O(\infty)\big)$ (formally), $\mathrm{Rep}\,\big(O(N)\big)$, $\mathrm{Rep}\,\big(Sp(N)\big)$ or of one of its associated fusion categories. If the braiding is not symmetric, they are completely determined by the eigenvalues of a certain braiding morphism, and we determine precisely which values can occur in the various cases. If the category allows a symmetric braiding, it is essentially determined by the dimension of the object corresponding to the vector representation.

1. INTRODUCTION

Braided tensor categories have played a prominent role in various areas in recent years, such as conformal field theory, string theory, operator algebras and low-dimensional topology. Important examples have been constructed in a mathematically rigorous way using the representation theory of quantum groups, loop groups and Kac-Moody algebras. This naturally leads to the question of classifying such categories. We solve this question in this paper for braided categories associated to the representation categories of orthogonal and symplectic groups, and various generalizations of them.

It has been shown in [23] that any rigid semisimple tensor category whose Grothendieck semiring is equivalent to the one of $\mathrm{Rep}\,\big(SU(N)\big)$ must necessarily be equivalent to the category $\mathrm{Rep}(U_q\,\mathfrak{sl}_N)$, with q not a root of unity, up to N possible choices of a twist; here $U_q\,\mathfrak{sl}_N$ is the Drinfeld-Jimbo q-deformation of the universal enveloping algebra $U\,\mathfrak{sl}_N$. The present paper proves a similar statement for a braided tensor category $\mathcal{C}$ whose Grothendieck semiring is isomorphic to the one of a full orthogonal or a symplectic group. It will be convenient to formulate the result in a slightly different way in this case: Let X be the object in $\mathcal{C}$ corresponding to the vector representation of an orthogonal or symplectic group. It is well-known that its second tensor power decomposes into the direct sum of three irreducible objects. Hence the braiding morphism $c_{X,X}$ has at most three different eigenvalues. It is easy to see that one can also define braiding structures for $\mathcal{C}$ by replacing $c_{X,X}$ by its inverse, its negative or its negative inverse. If $c_{X,X}$ has three distinct eigenvalues, $\mathcal{C}$ is completely classified as a monoidal category by these eigenvalues. Another set of eigenvalues belongs to a category equivalent to $\mathcal{C}$ if and only if it can be obtained from the ones of $c_{X,X}$ by changing the braiding structure as just mentioned before. Moreover, we also show that the eigenvalues have to be of the form $q, -q^{-1}$ and r^{-1}, or of the form $iq, -iq^{-1}$ and ir^{-1}, with q not a root of unity and with r being $\pm$ a power of q, where the exponent depends on the particular orthogonal or symplectic group. Here the two possible forms of the eigenvalues correspond to the two possible twists (in the language of [23]) for categories of this type. If $c_{X,X}$ has only two distinct eigenvalues, they are necessarily of the form $\{\pm 1\}$ or $\{\pm i\}$, and the category is completely determined by this and the quantity $d(X)$, which, up to a sign, is equal to the categorical dimension of the object X. In particular, we obtain two distinct families of categories whose Grothendieck semirings are isomorphic to the one of an odd-dimensional orthogonal group, while there is only one such family

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if the Grothendieck semiring is the one of an even-dimensional orthogonal, a symplectic or a special unitary group (see Cor 9.5 for a more precise statement).

It is easy to define a Grothendieck semiring which could be considered as the one of a formal group $O(\infty)$, and one can define categories with such a Grothendieck semiring. The methods in our paper apply similarly to classify such categories, and we obtain essentially the same classification as in the last paragraph. The only difference is that now r can not be $\pm$ a power of q, and q can not be a root of unity. Finally, our methods also apply to fusion categories whose Grothendieck semirings are quotients of the ones of an orthogonal or symplectic group. Here q has to be a root of unity and r is a power of q, where the order of the root of unity and the exponent depend on the given Grothendieck semiring. We also remark that in our context the braiding condition is strong enough that we never need to consider the full Grothendieck semiring; it suffices to know how to tensor with the vector representation.

The method of proof in this paper is similar to the one in [23]. We first give an intrinsic description of the endomorphisms of tensor powers of an object X corresponding to the vector representation of an orthogonal or symplectic group in terms of certain representations of braid groups. From this one can reconstruct the whole category, similarly as it was done in [23]. In this paper, we do this following an alternate approach due to Alain Bruguières. Besides that, the main differences to the paper [23] are that we have to assume a priori that these categories are braided (which may not be necessary) and that the braid representations as well as the combinatorics involved here are more complicated than the ones in [23].

Here are the contents of this paper in more detail. We first recall basic definitions of braided rigid tensor categories. We then present reconstruction techniques of [23] and from Bruguières' unpublished lecture notes [9]; in particular, Section 4 closely follows these notes. In Section 6, we derive relations for the braid representations occurring in $\mathrm{End}\left(X^{\otimes n}\right)$. We then study the corresponding abstract algebras given by these relations, which depend on two parameters. The main difficulty then is to show that these algebras map surjectively onto $\mathrm{End}\left(X^{\otimes n}\right)$. Here the crucial idea is, as in [23], the abstract characterization of the trace functional on $\mathrm{End}\left(X^{\otimes n}\right)$ coming from the dimension function as a so-called Markov trace. This shows that the image has to contain at least the quotient of this algebra modulo the annihilator ideal of the Markov trace. Rigidity is then used to prove that the image actually has to be equal to the quotient. This result together with the reconstruction results in Sections 3 and 4 is then used to prove the classification result in the last section.

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2. Definitions and notation

We recall some basic definitions and set up notations. For more details, we refer to [27], [13] for general categorical notions, and to [18], [37] for tensor categories.

Definition 2.1. *A* monoidal category $\mathcal{C}$ *is a category* $\mathcal{C}$ *with a functor* $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *called the* tensor product*, a natural isomorphism* a *between* $\otimes \circ (\otimes \times 1_{\mathcal{C}})$ *and* $\otimes \circ (1_{\mathcal{C}} \times \otimes)$ *called the* associativity constraint*, satisfying the* pentagon axiom*, a unit object* $\mathbb{1} \in \mathcal{C}$ *and natural isomorphisms* $l_X : \mathbb{1} \otimes X \to X$ *and* $r_X : X \otimes \mathbb{1} \to X$ *called the left and right* unit constraints *satisfying the* triangle axiom.

The pentagon axiom just states that different ways of rebracketing the tensor product of four objects will lead to the same results, see e.g. [18] for a precise statement. The triangle axiom just states that the left and right constraints are compatible with associativity, i.e. that $(1_X \otimes l_Y) \circ a_{X,\mathbf{1},Y}$ and $r_X \otimes 1_Y$ describe the same morphism from $(X \otimes \mathbb{1}) \otimes Y$ to $X \otimes Y$; here $a_{X,\mathbf{1},Y}$ is the associativity morphism $(X \otimes \mathbb{1}) \otimes Y \to X \otimes (\mathbb{1} \otimes Y)$. A *monoidal functor* is a triple (F, θ, ϕ), where $F : \mathcal{C} \to \mathcal{C}'$ is a functor, $\theta \in \mathrm{Hom}_{\mathcal{C}'}\big(F(\mathbb{1}), \mathbb{1}'\big)$ is an isomorphism and ϕ is a natural isomorphism

$$\phi_{X,Y} : F(X) \otimes' F(Y) \to F(X \otimes Y).$$

In order to respect the monoidal structure, θ and ϕ and required to satisfy certain obvious commutative diagrams. See e.g. [18], Ch. XI.4 for the full definition.

A monoidal category $\mathcal{C}$ is called strict if a, l, and r are the identity. That is $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ and $\mathbb{1} \otimes X = X \otimes \mathbb{1} = X$ for any $X \in \mathcal{C}$. A theorem of Mac Lane's asserts that any monoidal category is equivalent to a strict one (see e.g. [18], p. 288). Since our interest is in characterizing tensor categories up to equivalence, we may and will assume our categories to be strict monoidal for the rest of the paper.

A strict monoidal category $\mathcal{C}$ is called *right rigid* if every object $X \in \mathcal{C}$ has a right dual object $X^* \in \mathcal{C}$ and a pair of morphisms $i_X : \mathbb{1} \to X \otimes X^*$ and $d_X : X^* \otimes X \to \mathbb{1}$ such that the maps

$$X = \mathbb{1} \otimes X \xrightarrow{i_X \otimes 1_X} X \otimes X^* \otimes X \xrightarrow{1_X \otimes d_X} X \otimes \mathbb{1} = X$$

$$X^* = X^* \otimes \mathbb{1} \xrightarrow{1_{X^*} \otimes i_X} X^* \otimes X \otimes X^* \xrightarrow{d_X \otimes 1_{X^*}} \mathbb{1} \otimes X^* = X^*$$

are 1_X and 1_{X^*}.

With this notion of duality, we also have the usual isomorphism between $\mathrm{Hom}(V, W \otimes X^*)$ and $\mathrm{Hom}(V \otimes X, W)$ for any objects V, W in $\mathcal{C}$. One checks easily that these isomorphisms are given by the maps $a \to (1_W \otimes d_X) \circ (a \otimes 1_X)$ and $b \to (b \otimes 1_{X^*}) \circ (1_V \otimes i_X)$ for $a \in \mathrm{Hom}(V, W \otimes X^*)$ and $b \in \mathrm{Hom}(V \otimes X, W)$. In particular, one obtains as a special case that $\dim \mathrm{Hom}(\mathbb{1}, X \otimes X^*) = \dim \mathrm{End}(X) = 1$ if X is a simple object. Left rigidity is defined similarly as right rigidity with the left dual ${}^*X$ of X on the opposite side of X.

A *tensor category* is an abelian category with the additional structure of a monoidal category such that the tensor product and the direct sum are distributive.

Definition 2.2. *A* $\mathbb{C}$-category $\mathcal{C}$ *is an additive category in which the morphisms between any two objects form a finite dimensional* $\mathbb{C}$*-vector space and composition of morphisms is bilinear relative to the vector space structure. A tensor category which is also a* $\mathbb{C}$*-category will be called a* $\mathbb{C}$-tensor category. *In this case, we will require that the categorical tensor be* $\mathbb{C}$*-bilinear.*

A strict monoidal category $\mathcal{C}$ is called *braided* if there exists a natural isomorphism $c_{X,Y} : X \otimes Y \to Y \otimes X$ called the *braiding* such that:

$$\begin{tikzcd} X \otimes Y \otimes Z \arrow[rr, "c_{X,Y\otimes Z}"] \arrow[dr, "c_{X,Y} \otimes 1_Z"'] & & Y \otimes Z \otimes X \\ & Y \otimes X \otimes Z \arrow[ur, "1_Y \otimes c_{X,Z}"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} X \otimes Y \otimes Z \arrow[rr, "c_{X\otimes Y,Z}"] \arrow[dr, "1_X \otimes c_{Y,Z}"'] & & Z \otimes X \otimes Y \\ & X \otimes Z \otimes Y \arrow[ur, "c_{X,Z} \otimes 1_Y"'] & \end{tikzcd}$$

are commuting diagrams. Naturality means that for any morphisms $f : X \to X'$ and $g : Y \to Y'$

$$(g \otimes f) \circ c_{X,Y} = c_{X',Y'} \circ (f \otimes g).$$

Let $\mathcal{C}$ and $\mathcal{C}'$ be strict braided monoidal categories. A monoidal functor (F, θ, ϕ) is called *braided* if it respects the braiding axioms in the sense that

$$F(c_{X,Y})\, \phi_{X,Y} = \phi_{Y,X}\, c'_{F(X),F(Y)}.$$

A braiding is a generalization of the flip, which is the natural isomorphism $P_{A,B} : A \otimes B \to B \otimes A$ on the category of modules over the commutative ring R. Note that the flip is involutive, that is $P_{B,A} \circ P_{A,B} = 1_{A \otimes B}$. This is not required for a braiding, but the property is generalized in the notion of the *twist*, which is a natural isomorphism $\theta_V : V \to V$ in a braided monoidal category $\mathcal{C}$ such that

$$\theta_{X \otimes Y} = c_{Y,X} \circ c_{X,Y} \circ (\theta_X \otimes \theta_Y)$$

for all $X, Y \in \mathcal{C}$. θ is required to be functorial in the sense that for any morphism $f : X \to Y$, $\theta_Y \circ f = f \circ \theta_X$. A *ribbon category* $\mathcal{C}$ is a rigid braided monoidal category with a compatible twist, meaning:

$$\big(\theta_X \otimes 1_{X^*}\big) \circ i_X = \big(1_X \otimes \theta_{X^*}\big) \circ i_X.$$

In a ribbon category, right rigidity also implies left rigidity and vice versa. In fact, given the right duality morphisms i and d,

$$i'_X = (1_{X^*} \otimes \theta_X) \circ c_{X,X^*} \circ i_X \quad \text{and} \quad d'_X = d_X \circ c_{X,X^*} \circ (\theta_X \otimes 1_{X^*}) \tag{2.1}$$

yield left duality morphisms which make the category left rigid. With this left duality, the left and right duals of objects and morphisms coincide.

We will also need the morphism

$$\tilde{e}_X = i'_X \circ d_X = i_X \circ d'_X \in \mathrm{End}\big(X \otimes X^*\big). \tag{2.2}$$

These allow us to define the *categorical trace* of an endomorphism $f \in \mathrm{End}(X)$ as

$$\mathrm{Tr}_X(f) = d'_X \circ (f \otimes 1_{X^*}) \circ i_X = d_X \circ c_{X,X^*} \circ \big((\theta_X \circ f) \otimes 1_{X^*}\big) \circ i_X \in \mathrm{End}(\mathbb{1}),$$

which is easily seen to be the same as

$$\mathrm{Tr}_X(f) = d_X \circ (1_{X^*} \otimes f) \circ i'_X = d_X \circ \big(1_{X^*} \otimes (\theta_X \circ f)\big) \circ c_{X,X^*} \circ i_X \in \mathrm{End}(\mathbb{1}),$$

using naturality of the braiding and the twist. Just like the usual trace of linear operators, $\mathrm{Tr}_Y(fg) = \mathrm{Tr}_X(gf)$ for any $f \in \mathrm{Hom}(X,Y)$ and $g \in \mathrm{Hom}(Y,X)$, and $\mathrm{Tr}_{X \otimes Y}(f \otimes g) = \mathrm{Tr}_X(f)\,\mathrm{Tr}_Y(g)$ for any $f \in \mathrm{End}(X)$ and $g \in \mathrm{End}(Y)$ (see [18] or [37] for a proof). If $f \in \mathrm{End}(\mathbb{1})$, then $\mathrm{Tr}_{\mathbb{1}}(f) = f$. The *categorical dimension* of an object X is $\dim X = \mathrm{Tr}_X\big(1_X\big)$. It is clear from the properties of the trace that $\dim X \oplus Y = \dim X + \dim Y$ and $\dim X \otimes Y = (\dim X)(\dim Y)$.

The *normalized trace* tr_X on $\mathrm{End}(X)$ is defined by $\mathrm{tr}_X(f) = \mathrm{Tr}_X(f)/(\dim X)$. In the following we will often just write Tr, tr for the trace or normalized trace when it is clear for which object it is defined.

We call a morphism a monomorphism or monic if its kernel is 0 and an epimorphism or epic if its cokernel is 0. As is customary, we won't get hung up on abusing the language slightly and calling object A a "subobject" of B if there exists a monomorphism $A \to B$, and referring to a monomorphism in the kernel of f as "a kernel."

3. Categorical reconstruction

In the following we will say that a $\mathbb{C}$-category $\mathcal{C}$ is *semisimple* if every endomorphism ring in $\mathcal{C}$ is a semisimple $\mathbb{C}$-algebra. An object Y in $\mathcal{C}$ is called *simple* if $\mathrm{End}(X) = \mathbb{C}$. This is a somewhat weaker definition of semisimplicity as is usually common, as can be seen at the following example.

Definition 3.1. *A* monoidal algebra $\mathcal{A}$ *is a semisimple monoidal category whose objects are the natural numbers with ordinary addition as the tensor product.*

To get an example of a monoidal algebra, let $\mathcal{C}$ be a semisimple monoidal category, and let X be an object in $\mathcal{C}$. Then the subcategory $\mathcal{A}$ whose objects are tensor powers of X (with the obvious labeling $X^{\otimes n} \longleftrightarrow n \in \mathbb{N}$) is a monoidal algebra; here we define $X^{\otimes 0} = \mathbb{1}$, the trivial object. It is well-known that if one takes for X the vector representation of a classical Lie group, the only simple objects in the corresponding monoidal algebra would be $\mathbb{1}$ and X itself.

However, it is well-known that the representation category of a classical Lie group is essentially determined if one understands the decomposition of tensor powers of its vector representation. This statement will be made precise and proved in this and the following section for general monoidal semisimple $\mathbb{C}$-tensor categories.

Let $\mathcal{C}$ be a monoidal category. In order to get direct sums (i.e. an additive category), we first define a larger category $\operatorname{Add}\mathcal{C}$ whose objects are finite sequences of objects from $\mathcal{C}$ including the empty sequence. The morphisms from $\big(X_1, X_2, \ldots, X_n\big)$ to $\big(Y_1, Y_2, \ldots, Y_m\big)$ are defined by

$$\operatorname{Hom}_{\operatorname{Add}\mathcal{C}}\Big(\big(X_1, X_2, \ldots, X_n\big),\ \big(Y_1, Y_2, \ldots, Y_m\big)\Big) = \bigoplus_{i,j} \operatorname{Hom}_{\mathcal{C}}\left(X_i, Y_j\right)$$

where $\oplus$ on the right-hand side stands for the ordinary direct sum of vector spaces. If either of the two sequences is empty, the Hom space will be the 0-vector space. We will compose morphisms, when possible, by ordinary matrix multiplication. We claim that this is an additive category with concatenation of sequences as the direct sum operation. The required direct sum system

$$\big(X_1, X_2, \ldots, X_n\big) \underset{\pi_1}{\overset{\iota_1}{\rightleftarrows}} \big(X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m\big) \underset{\pi_2}{\overset{\iota_2}{\leftrightarrows}} \big(Y_1, Y_2, \ldots, Y_m\big)$$

is constructed the obvious way from identities in $\operatorname{End}\big(X_i\big)$ and $\operatorname{End}\big(Y_j\big)$ and zeros in the other components.

We still need to get enough subobjects. This will be accomplished by a process called the idempotent completion (see [13], p. 61), which goes as follows. Starting with any category $\mathcal{C}$, let the objects of $\operatorname{Idem}\mathcal{C}$ be the pairs (X, p) where $X \in \operatorname{Ob}(\mathcal{C})$ and $p \in \operatorname{End}(X)$ with $p^2 = p$, that is p is an idempotent. The morphisms in $\operatorname{Idem}\mathcal{C}$ are defined as follows

$$\operatorname{Hom}_{\operatorname{Idem}\mathcal{C}}\big((X,p),(Y,q)\big) = \big\{f \in \operatorname{Hom}_{\operatorname{Add}\mathcal{C}}(X,Y) \mid f = qf = fp\big\}.$$

We will say the idempotent p splits if it can be factored as $p = ab$ with a monic and b epic. In this case, it it easy to see $ba = 1$ (identity of the source of a) by canceling a on the left and b on the right from $ab = p = p^2 = abab$. It is an easy exercise to show that idempotents split in $\operatorname{Idem}\mathcal{C}$.

Before we prove that these constructions indeed produce an abelian category, we will need a lemma about the existence of quasi-inverses.

Lemma 3.2. *Let $\mathcal{C}$ be a semisimple additive $\mathbb{C}$-category and $f \in \operatorname{Hom}(X,Y)$ for some objects X, Y. Then there exists $g \in \operatorname{Hom}(Y,X)$ with $f = fgf$ and with $P = fg$ and $Q = gf$ idempotents in $\operatorname{End}(Y)$ and $\operatorname{End}(X)$ respectively. If f is monic, then $Q = 1_X$ and P splits as fg. If f is epic, then $P = 1_Y$ and Q splits as gf.*

Proof: We can naturally embed $\operatorname{Hom}(X,Y)$ and $\operatorname{Hom}(Y,X)$ into $\operatorname{End}(X \oplus Y)$. Hence we can consider f as an element in $\operatorname{End}(X \oplus Y)$, which is semisimple. Restricting to a simple component, it suffices to consider $f' \in M_n(\mathbb{C})$, acting on $V = \mathbb{C}^n$. Let $V_1 = \ker(f')$ and W_1 such that $V_1 \oplus W_1 = V$. Let $W_2 = \operatorname{Im}(f')$ and V_2 such that $V_2 \oplus W_2 = V$. Hence $f'|_{W_1} : W_1 \to W_2$ is an isomorphism. Let $g' : W_2 \to W_1$ be the inverse of f. Extend g' to V by letting it act as 0 on V_2. Doing this for each simple component of $\operatorname{End}(X \oplus Y)$, we obtain an element $\tilde{g} \in \operatorname{End}(X \oplus Y)$ such that $f = f\tilde{g}f$. Then $g = \pi_1 \tilde{g} \iota_2 \in \operatorname{Hom}(Y,X)$ satisfies $fgf = f$, where

$$X \underset{\pi_1}{\overset{\iota_1}{\rightleftarrows}} X \oplus Y \underset{\pi_2}{\overset{\iota_2}{\leftrightarrows}} Y$$

is a direct sum system in $\mathcal{C}$

That $P^2 = P$ and $Q^2 = Q$ is trivial. If f is monic, cancel f on the left from $f = fgf$ to get $1_Y = gf$, which makes g necessarily epic, hence P splits as fg and similarly for f epic. □

Theorem 3.3. *Let $\mathcal{C}$ be a semisimple $\mathbb{C}$-category. Then $\operatorname{Ab}\mathcal{C} = \operatorname{Idem}\operatorname{Add}\mathcal{C}$ is a semisimple abelian category .*

Proof: The fact that $\mathrm{Ab}\,\mathcal{C}$ has direct sums (i.e. it is an additive category) follows easily by applying the construction at the beginning of this section to objects of $\mathrm{Idem}\,\mathrm{Add}\,\mathcal{C}$. This is left to the reader. To show that $\mathrm{Ab}\,\mathcal{C}$ is also abelian, we need to check

1.Every morphism $f \in \mathrm{Hom}\big((X,p),(Y,q)\big)$ must have a kernel and a cokernel. Let us construct a kernel. Let

$$I = \big\{g \in \mathrm{End}(X,p) \mid fg = 0\big\}.$$

Clearly, I is a right ideal of $\mathrm{End}(X,p)$, hence $I = P\ \mathrm{End}(X,p)$ for some idempotent $P \in \mathrm{End}(X,p)$ by semisimplicity. We would like to claim that $P : (X,P) \to (X,p)$ is a kernel of f. P is monic by definition: if $P\alpha_1 = P\alpha_2$ for some $\alpha_1, \alpha_2 \in \mathrm{Hom}\big((Z,r),(X,P)\big)$ then

$$\alpha_1 = P\alpha_1 = P\alpha_2 = \alpha_2.$$

That $fP = 0$ is clear. Now, suppose $fg = 0$ for some $g \in \mathrm{Hom}\big((Z,r),(X,p)\big)$. We will show g factors through P. By Lemma 3.2, we have $h \in \mathrm{Hom}\big((X,p),(Z,r)\big)$ such that $g = ghg$. Now $f(gh) = (fg)h = 0$, hence $gh \in I = P\ \mathrm{End}(X,p)$. Thus

$$Pg = P(ghg) = (Pgh)g = (gh)g = g.$$

The dual construction will give a cokernel of f.

2. We need to show that every monomorphism is a kernel and every epimorphism is a cokernel. Let $f \in \mathrm{Hom}\big((X,p),(Y,q)\big)$ be a monomorphism. Invoke Lemma 3.2 again to find $g \in \mathrm{Hom}\big((Y,q),(X,p)\big)$ such that $f = fgf$. Let $P = 1 - fg \in End(Y,q)$. Clearly, $Pf = 0$. If $h \in \mathrm{Hom}\big((Z,r),(Y,q)\big)$ such that $Ph = 0$, then $h = fgh$, hence h factors through f. As f is already monic, f is a kernel of P. The dual argument shows that every epimorphism is a cokernel.

That $\mathrm{Ab}\,\mathcal{C}$ is semisimple is clear as its endomorphism rings are subalgebras of the endomorphism rings in $\mathrm{Add}\,\mathcal{C}$, which are semisimple. □

If $\mathcal{C}$ is a monoidal category to begin with the tensor functor $\otimes$ on $\mathcal{C}$ is extended to a tensor product $\otimes_{\mathrm{Ab}\,\mathcal{C}}$ in the resulting abelian category $\mathrm{Ab}\,\mathcal{C}$ in the obvious way as follows: In $\mathrm{Add}\,\mathcal{C}$, define $\otimes_{\mathrm{Add}\,\mathcal{C}}$ as

$$\big(X_1 \oplus X_2 \oplus \cdots \oplus X_n\big) \otimes_{\mathrm{Add}\,\mathcal{C}} \big(Y_1 \oplus Y_2 \oplus \cdots \oplus Y_m\big) = \bigoplus_{i,j} \big(X_i \otimes_{\mathcal{C}} Y_j\big)$$

on the objects and analogously on the morphisms. (Where $\oplus$ is the categorical direct sum constructed previously.) In $\mathrm{Ab}\,\mathcal{C}$, define $\otimes_{\mathrm{Ab}\,\mathcal{C}}$ as

$$(X,p) \otimes_{\mathrm{Ab}\,\mathcal{C}} (Y,q) = \big(X \otimes_{\mathrm{Add}\,\mathcal{C}} Y, p \otimes_{\mathrm{Add}\,\mathcal{C}} q\big)$$

on the objects and simply as $\otimes_{\mathrm{Add}\,\mathcal{C}}$ on the morphisms.

We also observe that if $\mathcal{D}$ is a full subcategory of a semisimple additive category $\mathcal{C}$, then $\mathrm{Add}\,\mathcal{D}$ is equivalent to the additive subcategory generated by $\mathcal{D}$ in $\mathcal{C}$, that is the full subcategory whose objects are finite direct sums of objects of $\mathcal{D}$ inside $\mathcal{C}$. We will in the following identify $\mathrm{Add}\,\mathcal{D}$ with that subcategory to simplify notation.

Theorem 3.4. *Let $\mathcal{C} = (\mathcal{C}, \otimes, a, \mathbb{1}, l, r)$ be a semisimple abelian $\mathbb{C}$-category and $\mathcal{D}$ a full subcategory (not necessarily abelian) of $\mathcal{C}$ that generates $\mathcal{C}$ in the sense that every object in $\mathcal{C}$ is a subobject of a direct sum of objects from $\mathcal{D}$. Then there is an equivalence of abelian categories:*

$$\mathrm{Ab}\,\mathcal{D} \cong \mathcal{C}.$$

Proof: We will construct the equivalence $F : \mathcal{C} \to \mathrm{Ab}\,\mathcal{D}$. Let $A \in \mathrm{Ob}(\mathcal{C})$. For every such object, we can choose $X_1, \dots X_n \in \mathrm{Ob}(\mathcal{D})$ and a monic $f : A \to X_1 \oplus \cdots \oplus X_n$ in $\mathcal{C}$ by the hypothesis. Use Lemma 3.2 in $\mathrm{Add}\,\mathcal{D}$ to find $g : X_1 \oplus \cdots \oplus X_n \to A$ such that $f = fgf$. As $fg \in \mathrm{End}\big(X_1 \oplus \cdots \oplus X_n\big)$ is an idempotent, we can set $F(A) = \big(X_1 \oplus \cdots \oplus X_n, fg\big)$.

Now, let $\sigma \in \mathrm{Hom}_{\mathcal{C}}(A,B)$. As above, there exist monomorphisms $f : A \to X_1 \oplus \cdots \oplus X_n$ and $h : B \to Y_1 \oplus \cdots \oplus Y_m$ in $\mathcal{C}$, and g and k such that $f = fgf$ and $h = hkh$. Then we already have

$F(A) = \big(X_1 \oplus \cdots \oplus X_n, fg\big)$ and $F(B) = \big(Y_1 \oplus \cdots \oplus Y_m, hk\big)$. Set $F(\sigma) = h\sigma g$. That this is indeed in $\mathrm{Hom}_{\mathrm{Ab}\,\mathcal{D}}\Big(\big(X_1 \oplus \cdots \oplus X_n, fg\big), \big(Y_1 \oplus \cdots \oplus Y_m, hk\big)\Big)$ follows from

$$h\sigma g = hk(h\sigma g) = (h\sigma g)fg.$$

F as a map $\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathrm{Ab}\,\mathcal{D}}\Big(\big(X_1 \oplus \cdots \oplus X_n, fg\big), \big(Y_1 \oplus \cdots \oplus Y_m, hk\big)\Big)$ in fact has an obvious inverse G that takes $\phi \in \mathrm{Hom}_{\mathrm{Ab}\,\mathcal{D}}\Big(\big(X_1 \oplus \cdots \oplus X_n, fg\big), \big(Y_1 \oplus \cdots \oplus Y_m, hk\big)\Big)$ to $k\phi f$.

We have just proven that F is full and faithful. It is now enough to show that each object in $\mathrm{Ab}\,\mathcal{D}$ is isomorphic to one in the image of F (see [27], p. 93) to conclude that F is an equivalence.

Let $\big(Y_1 \oplus \cdots \oplus Y_m, p\big)$ be an object in $\mathrm{Ab}\,\mathcal{D}$. Then p is an idempotent in $\mathrm{End}_{\mathcal{C}}\big(Y_1 \oplus \cdots \oplus Y_m\big)$. In an abelian category, every morphism has a factorization into an epimorphism followed by a monomorphism (see [27], p. 199). So in particular, idempotents split. Let p split as ab and set $A = S(a)$. Then $a : A \to Y_1 \oplus \cdots \oplus Y_m$ is a subobject, and we claim $F(A)$ is isomorphic to $\big(Y_1 \oplus \cdots \oplus Y_m, p\big)$. For suppose that in the construction of F above we chose the subobject $f : A \to X_1 \oplus \cdots \oplus X_n$ and $F(A) = \big(X_1 \oplus \cdots \oplus X_n, fg\big)$. Then it is easy to verify that ag is an isomorphism in

$$\mathrm{Hom}_{\mathrm{Ab}\,\mathcal{D}}\Big(\big(X_1 \oplus \cdots \oplus X_n, fg\big), \big(Y_1 \oplus \cdots \oplus Y_m, ab\big)\Big)$$

with inverse fb. $\square$

Note that we are making a lot of arbitrary choices in constructing this equivalence. This is to be expected though, as equivalences are usually not unique. Compare this with isomorphism between groups: one can normally find several different isomorphisms between two isomorphic groups.

In fact, a closer look at F reveals that if $\mathcal{C}$ is a monoidal category and $\mathcal{D}$ is a submonoidal category, then F extends to a monoidal functor. The proof is long and tedious, but is straightforward and merely an exercise in applying definitions, so we will omit it here. Hence F is an equivalence of tensor categories and we have

Theorem 3.5. *Let $\mathcal{C}$ be a semisimple tensor category and $\mathcal{D} \subseteq \mathcal{C}$ a full submonoidal category. Suppose that $\mathcal{D}$ generates $\mathcal{C}$ in the sense that every object in $\mathcal{C}$ is a subobject of a direct sum of objects from $\mathcal{D}$. Then there is an equivalence of tensor categories:*

$$\mathrm{Ab}\,\mathcal{D} \cong \mathcal{C}.$$

We will use this result in the following context: Let $\mathcal{C}$ be a semisimple tensor category, and let X be an object in $\mathcal{C}$ which generates $\mathcal{C}$ in the sense that every simple object of $\mathcal{C}$ is a subobject of some tensor power of X. Let $\mathcal{A}(\mathcal{C}, X)$ be the monoidal algebra generated by X, as described at the beginning of this section. Then the monoidal algebra $\mathcal{A}(\mathcal{C}, X)$ obviously inherits the braiding, and it is straightforward to show that the equivalence in the last theorem is an equivalence of braided categories. Hence we obtain

Corollary 3.6. *With the just introduced notations we have the equivalence of braided categories $\mathrm{Ab}(\mathcal{A}(\mathcal{C}, X)) \cong \mathcal{C}$.*

4. Extending diagonals of braided monoidal algebras

The results of this section have already apeared in [23]. Here we closely follow the presentation which was subsequently given by Bruguières in unpublished lecture notes [9] and which has some advantages over the original one in our context. We would like to thank Alain Bruguières for allowing us to include this material in our paper.

The precise goal of this section will be stated after Definition 4.3. In the following $\mathcal{C}$ is a semisimple (not necessarily braided) tensor category, X is an object in $\mathcal{C}$ and $\mathcal{A} = \mathcal{A}(\mathcal{C}, X)$ is the associated monoidal algebra, as in the last section.

Definition 4.1. *A monoidal algebra* $\mathcal{A} = \mathcal{A}(\mathcal{C}, X)$ *is* of type N *if*

(a) $\mathrm{Hom}_{\mathcal{A}}\left(X^{\otimes m}, X^{\otimes n}\right) = 0$ *unless* $m \equiv n \mod N$.
(b) $\mathbb{1}$ *and* X *are simple.*
(c) $\mathrm{Hom}_{\mathcal{A}}\left(\mathbb{1}, X^{\otimes N}\right) = \mathrm{Hom}_{\mathcal{A}}\left(X^{\otimes N}, \mathbb{1}\right) = \mathbb{C}$.

This, for example, holds for the monoidal algebra arising from the vector representation in the representation categories of $SU(N)$ and $U_q\,\mathfrak{sl}_N$, and also for orthogonal and symplectic categories with $N = 2$ (see Section 6).

Lemma 4.2. *Let* $\mathcal{A}$ *be a monoidal algebra of type* N.

(a) *There exist nonzero morphisms* $\iota : \mathbb{1} \to X^{\otimes N}$ *and* $\pi : X^{\otimes N} \to \mathbb{1}$ *such that* $\pi\iota = 1_{\mathbb{1}}$ *and* $\iota\pi = \Pi$ *is an idempotent in* $\mathrm{End}\left(X^{\otimes N}\right)$ *independent of the choices of* ι *and* π.
(b) $\dim\left\{f \in \mathrm{End}\left(X^{\otimes N}\right) \mid f\Pi = f = \Pi f\right\} = 1$.
(c) *For any* $n \in \mathbb{N}$, *the map* $\phi : \mathrm{End}\left(X^{\otimes n}\right) \to \mathrm{End}\left(X^{\otimes n+N}\right)$ *which takes* f *to* $f \otimes \Pi$ *is an isomorphism onto*

$$\Sigma = \left\{g \in \mathrm{End}\left(X^{\otimes n+N}\right) \middle| \left(1_{X^{\otimes n}} \otimes \Pi\right) g = g\left(1_{X^{\otimes n}} \otimes \Pi\right) = g\right\}.$$

Proof: Let $\iota : \mathbb{1} \to X^{\otimes N}$ be a nonzero morphism. By Lemma 3.2 there exists a morphism $\pi : X^{\otimes N} \to \mathbb{1}$ such that $\iota\pi\iota = \iota$. It follows that $\pi\iota = 1_{\mathbb{1}} = 1 \in \mathbb{C}$, and $\Pi = \iota\pi \in \mathrm{End}\left(X^{\otimes N}\right)$ is an idempotent. This idempotent is unique as the object $\mathbb{1}$ appears with multiplicity 1 in $X^{\otimes N}$.

The second statement is a consequence of the last statement with $n = 0$. To prove the last statement observe that $\phi(f) \in \Sigma$ is clear from the first property of Π. Let $\psi : \Sigma \to \mathrm{End}\left(X^{\otimes n}\right)$ be defined by

$$\psi(g) = \left(1_{X^{\otimes n}} \otimes \pi\right) g \left(1_{X^{\otimes n}} \otimes \iota\right).$$

Then it is straightforward to check that ψ is the inverse of ϕ, which finishes the proof of the lemma. □

Definition 4.3. *The* diagonal $\mathcal{D} = \Delta\mathcal{A}$ *of a monoidal algebra* $\mathcal{A}$ *is the monoidal algebra with*

$$\mathrm{Hom}_{\mathcal{D}}\left(X^{\otimes m}, X^{\otimes n}\right) = 0 \textit{ if } m \neq n$$

and

$$\mathrm{End}_{\mathcal{D}}\left(X^{\otimes n}\right) = \mathrm{End}_{\mathcal{A}}\left(X^{\otimes n}\right).$$

We will now investigate to what extent the structure of a monoidal algebra of type N can be recovered from its diagonal. So let $\mathcal{D}$ be a braided diagonal monoidal algebra with braiding c, which is the diagonal of a (not necessarily braided) monoidal algebra $\mathcal{A}$ of type N. We attach a complex number $\Theta(\mathcal{A})$ to $\mathcal{A}$ as follows:

$$\Theta(\mathcal{A}) = l_X\left(\pi \otimes 1_X\right) c_{1,N}\left(1_X \otimes \iota\right) r_X^{-1} \in \mathrm{End}(X) = \mathbb{C}. \tag{4.1}$$

In fact, since $\mathcal{A}$ is a strict category $l_X = r_X = 1_X$. So we are free to suppress them. We will simply denote $\Theta(\mathcal{A})$ by Θ whenever the context is clear. Observe that Θ, just like Π depends only on $\mathcal{A}$ and not on the particular choice of π and ι.

We will now prove some simple results for the braided diagonals of monoidal algebras $\mathcal{A} = \mathcal{A}(\mathcal{C}, X)$ of type N. To keep the notation from becoming overwhelming, we will use the simplified notation

$$c_{m,n} = c_{X^{\otimes m}, X^{\otimes n}}$$

for the braiding.

Lemma 4.4. *Let* $\mathcal{A}$ *be a monoidal algebra of type* N. *Suppose its diagonal* $\mathcal{D}$ *has a braiding* c. *Then we have*

(a) $\left(\pi \otimes 1_X\right) c_{1,N} = \Theta\left(1_X \otimes \pi\right)$ *and* $c_{1,N}\left(1_X \otimes \iota\right) = \Theta\left(\iota \otimes 1_X\right)$.

(b) $\big(1_X \otimes \pi\big)\, c_{N,1} = \Theta\big(\pi \otimes 1_X\big)$ *and* $c_{N,1}\,\big(\iota \otimes 1_X\big) = \Theta\big(1_X \otimes \iota\big)$.
(c) $c_{N,N}\,\big(\Pi \otimes \Pi\big) = \big(\Pi \otimes \Pi\big)\, c_{N,N} = \Pi \otimes \Pi$.

Proof: We will prove the first statement and leave the rest to the reader.

$$\begin{aligned}\big(\pi \otimes 1_X\big)\, c_{1,N} &= \big(\pi \otimes 1_X\big)\big(\iota \otimes 1_X\big)\big(\pi \otimes 1_X\big)\, c_{1,N}\\ &= \big(\pi \otimes 1_X\big)\, c_{1,N}\,\big(1_X \otimes \iota\big)\,\big(1_X \otimes \pi\big) = \Theta\big(1_X \otimes \pi\big).\end{aligned}$$

where the first equality holds because $\pi\iota = 1$, the second because $\iota\pi = \Pi \in \mathrm{End}\big(X^{\otimes N}\big)$ which is in $\mathcal{D}$ and c is functorial on $\mathcal{D}$, and the third is by the definition of Θ. The second part of the first statement goes similarly. □

Let $\mathcal{A}$ and $\mathcal{A}'$ be two monoidal algebras of type N with braided diagonals. We say that $\mathcal{A}$ and $\mathcal{A}'$ are extensions of the diagonal $\mathcal{D} = \Delta\mathcal{A}$ if there is an equivalence Ψ between $\mathcal{D}$ and the diagonal $\mathcal{D}'$ of $\mathcal{A}'$ as braided categories such that $\Psi(X^{\otimes n}) = (X')^{\otimes n}$ for all $n \in \mathbb{N}$. We say that the extensions $\mathcal{A}$ and $\mathcal{A}'$ of $\mathcal{D}$ are *diagonally equivalent* if Ψ can be extended to an equivalence $\Phi : \mathcal{A} \to \mathcal{A}'$ of monoidal algebras.

We are going to show that $\Theta(\mathcal{A})$ is an invariant under diagonal equivalence.

Proposition 4.5.
(a) *Let $\mathcal{A}$ and $\mathcal{A}'$ be monoidal algebras of type N and $\Phi : \mathcal{A} \to \mathcal{A}'$ a diagonal equivalence. Then $\Theta(\mathcal{A}) = \Theta(\mathcal{A}')$.*
(b) $(\Theta(\mathcal{A}))^N = 1$.

Proof: Since Φ is a monoidal functor $\mathcal{A} \to \mathcal{A}'$, it comes equipped with the isomorphism $\theta : \Phi(\mathbb{1}) \to \mathbb{1}'$ and the natural isomorphism

$$\phi_{i,j} : \Phi(X^{\otimes i}) \otimes' \Phi(X^{\otimes j}) \to \Phi(X^{\otimes i+j})$$

compatible with the action of Φ on morphisms (see e.g. [18], Ch. XI.4). This means, in particular, that we have for any morphisms $f : X^{\otimes i} \to X^{\otimes r}$ and $g : X^{\otimes j} \to X^{\otimes s}$

$$\Phi(f \otimes g) = \phi_{r,s}^{-1} \circ \big(\Phi(f) \otimes' \Phi(g)\big) \circ \phi_{i,j}$$

and compatibility with the braiding means that

$$c'_{i,j} = \phi_{j,i}^{-1} \circ \Phi(c_{i,j}) \circ \phi_{i,j}.$$

Moreover, compatibility with the left and right unit constraints translates into the identities

$$\Phi(l_X) \circ \phi_{0,1} = l'_{X'} \circ (\theta \otimes 1_{X'}) \qquad \text{and} \qquad \Phi(r_X) \circ \phi_{1,0} = r'_{X'} \circ (1_{X'} \otimes \theta).$$

But monoidal algebras are strict monoidal categories, so the unit constraints are identities. Using the bilinearity of the tensor product and the naturality of the unit constraints we obtain

$$\phi_{0,1} = \theta \otimes 1_{X'} = 1_{\mathbb{1}} \otimes \theta\, 1_{X'} = \theta\, 1_{X'} \qquad \text{and} \qquad \phi_{1,0} = 1_{X'} \otimes \theta = \theta\, 1_{X'} \otimes 1_{\mathbb{1}} = \theta\, 1_{X'},$$

and thus $\phi_{0,1} = \phi_{1,0}$. Now observe

$$\Phi(\pi_{\mathcal{A}})\, \Phi(\iota_{\mathcal{A}}) = \Phi(\pi_{\mathcal{A}} \iota_{\mathcal{A}}) = \Phi(1_{\mathbb{1}}) = 1_{\mathbb{1}}\,.$$

Hence we can and will choose $\pi_{\mathcal{A}'} = \Phi\left(\pi_{\mathcal{A}}\right)$ and $\iota_{\mathcal{A}'} = \Phi\left(\iota_{\mathcal{A}}\right)$. As we pointed out, $\Theta(\mathcal{A}')$ is independent of the particular choice of $\pi_{\mathcal{A}'}$ and $\iota_{\mathcal{A}'}$. Using this and the identities above, we obtain

$$\begin{aligned}\Theta(\mathcal{A}') &= \big(\pi_{\mathcal{A}'} \otimes' 1_{X'}\big)\, c'_{1,N}\,\big(1_{X'} \otimes' \iota_{\mathcal{A}'}\big)\\ &= \big(\phi_{0,1}^{-1} \circ \Phi(\pi_{\mathcal{A}} \otimes 1_X) \circ \phi_{N,1}\big)\big(\phi_{N,1}^{-1}\, \Phi(c_{1,N})\, \phi_{1,N}\big)\big(\phi_{1,N}^{-1} \circ \Phi(1_X \otimes \iota_{\mathcal{A}}) \circ \phi_{1,0}\big)\\ &= \phi_{0,1}^{-1}\, \Phi\big((\pi_{\mathcal{A}} \otimes 1_X)\, c_{1,N}\, (1_X \otimes \iota_{\mathcal{A}})\big)\, \phi_{1,0} = \phi_{0,1}^{-1}\, \Phi\big(\Theta(\mathcal{A})\big)\, \phi_{1,0}\\ &= \phi_{0,1}^{-1}\, \Theta(\mathcal{A})\, \phi_{1,0} = \Theta(\mathcal{A}),\end{aligned}$$

where $\Phi\big(\Theta(\mathcal{A})\big) = \Theta(\mathcal{A})$ because $\mathrm{End}_{\mathcal{A}}(X) = \mathrm{End}_{\mathcal{A}'}(X) = \mathbb{C}$ and $\Phi(1_X) = 1_{X'}$.

To prove the second statement, observe that $c_{n,N}(1_{X^{\otimes n}} \otimes \iota) = \Theta^n(\iota \otimes 1_{X^{\otimes n}})$; this follows from Lemma 4.4(a) by induction on n, using $c_{n,N} = (c_{1,N} \otimes 1_{X^{\otimes n-1}})(1_X \otimes c_{n-1,N})$. Hence we obtain, using Lemma 4.4(c),

$$\begin{aligned} \Pi \otimes \Pi &= c_{N,N}(1_{X^{\otimes n}} \otimes \iota)\iota(\pi \otimes \pi) \\ &= \Theta^N(\iota \otimes 1_{X^{\otimes n}})\iota(\pi \otimes \pi) \\ &= \Theta^N(\iota \otimes \iota)(\pi \otimes \pi) = \Theta^N(\Pi \otimes \Pi). \end{aligned}$$

□

Proposition 4.6. *Let $\mathcal{A}$ and $\mathcal{A}'$ be monoidal algebras of type N which are extensions of a given diagonal algebra $\mathcal{D}$. If $\Theta(\mathcal{A}) = \Theta(\mathcal{A}')$, then $\mathcal{A}$ and $\mathcal{A}'$ are diagonally equivalent.*

Proof: Choose $\iota_{\mathcal{A}}$, $\iota_{\mathcal{A}'}$, $\pi_{\mathcal{A}}$, and $\pi_{\mathcal{A}'}$ which satisfy the conditions of the morphisms ι and π in Lemma 4.2 for $\mathcal{A}$ and $\mathcal{A}'$. We will construct an equivalence $\Phi : \mathcal{A} \to \mathcal{A}'$ of monoidal algebras extending the equivalence Ψ between their diagonals. Define $\Phi\mid_{\mathcal{D}} = 1_{\mathcal{D}}$, $\Phi(\iota_{\mathcal{A}}) = \iota_{\mathcal{A}'}$ and $\Phi(\pi_{\mathcal{A}}) = \pi_{\mathcal{A}'}$. This will ensure uniqueness of a functor Φ.

If $m \equiv n \mod N$, let $p, \alpha, \beta \in \mathbb{N}$ such that $p = m + \alpha N = n + \beta N$. The idea is to pad f with ι's on the right and π's on the left so that the result is in $\mathrm{End}\left(X^{\otimes p}\right)$. Let

$$f_p = \left(1_{X^{\otimes n}} \otimes \iota^{\otimes \beta}\right) f \left(1_{X^{\otimes m}} \otimes \pi^{\otimes \alpha}\right) \in \mathrm{End}\left(X^{\otimes p}\right). \tag{4.2}$$

Note that f_p is a morphism in $\Delta\mathcal{A}$. Multiplying the last equation by $\left(1_{X^{\otimes n}} \otimes \pi^{\otimes \beta}\right)$ from the left and by $\left(1_{X^{\otimes m}} \otimes \iota^{\otimes \alpha}\right)$ from the right, we obtain

$$f = \left(1_{X^{\otimes n}} \otimes \pi^{\otimes \beta}\right) f_p \left(1_{X^{\otimes m}} \otimes \iota^{\otimes \alpha}\right). \tag{4.3}$$

As f_p is a morphism in $\Delta(\mathcal{A})$, we can define

$$\Phi(f) = \left(1_{X^{\otimes n}} \otimes \pi_{\mathcal{A}'}^{\otimes \beta}\right) \Psi(f_p) \left(1_{X^{\otimes m}} \otimes \iota_{\mathcal{A}'}^{\otimes \alpha}\right). \tag{4.4}$$

It is easy to check that $\Phi(f)$ does not depend on the choice of p. We still need to make sure that Φ is well-behaved with respect to the tensor product. Let $f \in \mathrm{Hom}_{\mathcal{A}}\left(X^{\otimes m}, X^{\otimes n}\right)$ and α, β, p such that $p = m + \alpha N = n + \beta N$. Let $g \in \mathrm{Hom}_{\mathcal{A}}\left(X^{\otimes m'}, X^{\otimes n'}\right)$ and α', β', p' such that $p' = m + \alpha' N = n + \beta' N$. Then

$$\begin{aligned} &\Phi(f) \otimes \Phi(g) = \\ &= \left(\left(1_{X^{\otimes n}} \otimes \pi_{\mathcal{A}'}^{\otimes \beta}\right) \Psi(f_p) \left(1_{X^{\otimes m}} \otimes \iota_{\mathcal{A}'}^{\otimes \alpha}\right)\right) \otimes \left(\left(1_{X^{\otimes n'}} \otimes \pi_{\mathcal{A}'}^{\otimes \beta'}\right) \Psi(g_{p'}) \left(1_{X^{\otimes m'}} \otimes \iota_{\mathcal{A}'}^{\otimes \alpha'}\right)\right) \\ &= \left(1_{X^{\otimes n}} \otimes \pi_{\mathcal{A}'}^{\otimes \beta} \otimes 1_{X^{\otimes n'}} \otimes \pi_{\mathcal{A}'}^{\otimes \beta'}\right) \Psi\big(f_p \otimes g_{p'}\big) \left(1_{X^{\otimes m}} \otimes \iota_{\mathcal{A}'}^{\otimes \alpha} \otimes 1_{X^{\otimes m'}} \otimes \iota_{\mathcal{A}'}^{\otimes \alpha'}\right). \end{aligned}$$

Now use Lemma 4.4 to move all the ι's and π's to the right in this last expression (remember to do so in $f_p \otimes g_{p'}$), and observe that all the Θ's and Θ^{-1}'s magically cancel. It is now clear that the expression we obtain is equal to $\Phi(f \otimes g)$. We can construct $\Phi^{-1} : \mathcal{A}' \to \mathcal{A}$ in the analogous way, which shows that Φ is indeed an equivalence of monoidal algebras. □

It follows from the last two propositions that there are at most N monoidal algebras of type N with the same diagonal. Before proving their existence, we need to determine the compatibility of their braidings.

Proposition 4.7. *Let c be a braiding on $\mathcal{D}$. Then c extends to a braiding on $\mathcal{A}$ if and only if $\Theta = 1$.*

Proof: $\Longrightarrow$: This is clear by functoriality.

$\Longleftarrow$: As c is a braiding on $\mathcal{D}$, it already satisfies most of the braiding axioms on $\mathcal{A}$ as well, except possibly functoriality. So all we have to prove is functoriality.

Now, let $f \in \mathrm{Hom}_{\mathcal{A}}\left(X^{\otimes m}, X^{\otimes n}\right)$. We will show $c_{1,n}\left(1_X \otimes f\right) = \left(f \otimes 1_X\right) c_{1,m}$. If $m \not\equiv n \mod N$, then $f = 0$ and the statement is obvious. Let $f \in \mathrm{Hom}_{\mathcal{A}}\left(X^{\otimes m}, X^{\otimes n}\right)$ and α, β, p as usual $p = m + \alpha N = n + \beta N$. Let f_p be as in Eq 4.2.

It follows from Lemma 4.4 (with $\Theta = 1$), the definition of $c_{n,m}$ and from $n + \beta N = p = m + \alpha N$ that

$$c_{1,n}\left(1_{X^{\otimes n+1}} \otimes \pi^{\otimes \beta}\right) = \left(1_{X^{\otimes n}} \otimes \pi^{\otimes \beta} \otimes 1_X\right) c_{1,p}.$$
$$c_{1,p}\left(1_{X^{\otimes m+1}} \otimes \iota^{\otimes \alpha}\right) = \left(1_{X^{\otimes m}} \otimes \iota^{\otimes \alpha} \otimes 1_X\right) c_{1,m}.$$

Using this and Eq 4.3 we obtain

$$\begin{aligned}
& c_{1,n}\left(1_X \otimes f\right) = \\
= \ & c_{1,n}\left(1_{X^{\otimes n+1}} \otimes \pi^{\otimes \beta}\right)\left(1_X \otimes f_p\right)\left(1_{X^{\otimes m+1}} \otimes \iota^{\otimes \alpha}\right) \\
= \ & \left(1_{X^{\otimes n}} \otimes \pi^{\otimes \beta} \otimes 1_X\right) c_{1,p}\left(1_X \otimes f_p\right)\left(1_{X^{\otimes m+1}} \otimes \iota^{\otimes \alpha}\right) \\
= \ & \left(1_{X^{\otimes n}} \otimes \pi^{\otimes \beta} \otimes 1_X\right)\left(f_p \otimes 1_X\right) c_{1,p}\left(1_{X^{\otimes m+1}} \otimes \iota^{\otimes \alpha}\right) \\
= \ & \left(f \otimes 1_X\right) c_{1,m}.
\end{aligned}$$

For $g \in \mathrm{Hom}_{\mathcal{A}}\left(X^{\otimes m'}, X^{\otimes n'}\right)$, a similar computation proves $c_{n',1}\left(g \otimes 1_X\right) = \left(1_X \otimes g\right) c_{m',1}$. Now we use induction to conclude

$$\begin{aligned}
& c_{n',n}(g \otimes f) = \\
= \ & c_{n',n}\left(1_{X^{\otimes n'}} \otimes f\right)\left(g \otimes 1_{X^{\otimes m}}\right) = \\
= \ & \left(f \otimes 1_{X^{\otimes n'}}\right) c_{n',m}\left(g \otimes 1_{X^{\otimes m}}\right) \\
= \ & \left(f \otimes 1_{X^{\otimes n'}}\right)\left(1_{X^{\otimes m}} \otimes g\right) c_{m',m} = (f \otimes g)\, c_{m',m}.
\end{aligned}$$

□

We can now prove the main result of this section. It first appeared in [23], with the presentation in this section following the notes [9] by Bruguières.

Theorem 4.8. *Let $\mathcal{D}$ be the diagonal of a braided monoidal algebra of type N. Then there exist exactly N monoidal algebras $\mathcal{A}$ such that $\mathcal{D} = \Delta(\mathcal{A})$ up to diagonal equivalence, one for each possible value of $\Theta(\mathcal{A})$.*

Proof: In view of our previous results, it suffices to construct a monoidal algebra $\mathcal{A}$ of type N such that $\Theta(\mathcal{A}) = \mu$ for each given N-th root of unity μ. Choose τ such that $\tau^N = 1/\mu$. Let $c'_{m,n} = \tau^{mn} c_{m,n}$. It is easy to see that this is still a braiding on $\mathcal{D}$. Denote the objects of $\mathcal{A}$ by $X^{\otimes n}$ as before. Let

$$\mathrm{Hom}_{\mathcal{A}}\left(X^{\otimes m}, X^{\otimes n}\right) = 0 \quad \text{if } m \not\equiv n \mod N,$$

otherwise let $p = m + \alpha N = n + \beta N$ $(\alpha, \beta \in \mathbb{N})$ and define

$$A_m^n(p) = \left\{ f \in \mathrm{End}_{\mathcal{D}}\left(X^{\otimes p}\right) \mid \left(1_{X^{\otimes n}} \otimes \Pi^{\otimes \beta}\right) f = f = f\left(1_{X^{\otimes m}} \otimes \Pi^{\otimes \alpha}\right) \right\}$$

Let $A_m^n = A_m^n(p)$ with $p = \max(m, n)$. By the 3rd property of Π, we know that the map $\phi : f \mapsto f \otimes \Pi$ is an injection $\mathrm{End}_{\mathcal{D}}\left(X^{\otimes p}\right) \to \mathrm{End}_{\mathcal{D}}\left(X^{\otimes p+N}\right)$. Observe that the restriction of ϕ to $A_m^n(p)$ has exactly $A_m^n(p+N)$ for its image in $\mathrm{End}_{\mathcal{D}}\left(X^{\otimes p+N}\right)$. Hence tensoring repeatedly on the right by Π gives us a chain of isomorphisms

$$A_m^n = A_m^n(p) \cong A_m^n(p+N) \cong A_m^n(p+2N) \cong \ldots.$$

Let $\phi_P : A_m^n \to A_m^n(P)$ be the induced isomorphism, with $P \equiv p \mod N$. Set

$$\mathrm{Hom}_{\mathcal{A}}\left(X^{\otimes m}, X^{\otimes n}\right) = A_m^n \cong A_m^n(p) \cong A_m^n(p+N) \cong \ldots.$$

In the following we will use these isomorphisms to define composition and tensor products for morphisms in $\mathcal{A}$. Let $g \in \operatorname{Hom}_{\mathcal{A}}\left(X^{\otimes k}, X^{\otimes m}\right)$ and $f \in \operatorname{Hom}_{\mathcal{A}}\left(X^{\otimes m}, X^{\otimes n}\right)$ with $k \equiv m \equiv n \mod N$. Choose some $P \geq \max(k, m, n)$ with

$$P = k + \alpha N = m + \beta N = n + \gamma N \qquad \alpha, \beta, \gamma \in \mathbb{N}.$$

Then we define $f \circ g$ by

$$f \circ g = \phi_P^{-1}(\phi_P(f) \circ \phi_P(g))$$

where the three ϕ_P's are three different maps and are to be understood in the appropriate context.

It is easy to see that this definition is independent of the choice of P. As the actual composing of maps always happens inside some $\operatorname{End}_{\mathcal{D}}\left(X^{\otimes P}\right)$, associativity of this composition law is inherited from $\mathcal{D}$.

We need to define a tensor product on this category. Let $f \in \operatorname{Hom}_{\mathcal{A}}\left(X^{\otimes m}, X^{\otimes n}\right)$ and $g \in \operatorname{Hom}_{\mathcal{A}}\left(X^{\otimes m'}, X^{\otimes n'}\right)$. Find p and p' such that $f \in A_m^n(p)$ and $g \in A_{m'}^{n'}(p')$ and

$$p = m + \alpha N = n + \beta N \text{ and } p' = m' + \alpha' N = n' + \beta' N.$$

Then

$$\left(1_{X^{\otimes n}} \otimes c'^{-1}_{n',\beta N} \otimes 1_{X^{\otimes \beta' N}}\right) \left(f \otimes_{\mathcal{D}} g\right) \left(1_{X^{\otimes m}} \otimes c'_{m',\alpha N} \otimes 1_{X^{\otimes \alpha' N}}\right)$$

is in $A_{m+m'}^{n+n'}(p+p')$. Applying $\phi_{p+p'}^{-1}$ to it gives us the desired morphism $f \otimes_{\mathcal{A}} g \in A_{m+m'}^{n+n'}$. That this is strictly associative follows from the strictness of the tensor product in $\mathcal{D}$ and the braiding axioms.

For $\mathcal{A}$ to be a monoidal algebra, it also needs to be a semisimple category, but that is obvious as the endomorphism rings of $\mathcal{A}$ all come from $\mathcal{D}$, which is already a monoidal algebra, hence semisimple. As

$$\operatorname{Hom}_{\mathcal{A}}\left(\mathbb{1}, X^{\otimes N}\right) \cong A_0^N(N) = \left\{f \in \operatorname{End}_{\mathcal{D}}\left(X^{\otimes N}\right) \mid \Pi f = f = f \Pi\right\} = \mathbb{C}$$

and similarly for $\operatorname{Hom}_{\mathcal{A}}\left(X^{\otimes N}, \mathbb{1}\right)$, $\mathcal{A}$ satisfies all of the conditions for being a monoidal algebra of type N. For ι and π in $\mathcal{A}$, choose Π considered as an element in $A_0^N(N)$ and as an element in $A_N^0(N)$ respectively. Then $\Pi(\mathcal{A}) = \pi\iota = \Pi^2 = \Pi$ by the 1st property of Π. We can now verify

$$\begin{aligned}
\Theta(\mathcal{A}) &= \left(\pi \otimes_{\mathcal{A}} 1_X\right) c_{1,N} \left(1_X \otimes_{\mathcal{A}} \iota\right) = \left(\underbrace{\Pi}_{\in A_N^0} \otimes_{\mathcal{A}} 1_X\right) c_{1,N} \left(1_X \otimes_{\mathcal{A}} \underbrace{\Pi}_{\in A_0^N}\right) \\
&= c'^{-1}_{1,N} \left(\Pi \otimes_{\mathcal{D}} 1_X\right) c_{1,N} \left(1_X \otimes_{\mathcal{D}} \Pi\right) \\
&= \tau^{-N} c_{1,N}^{-1} c_{1,N} \left(\underbrace{1_X \otimes_{\mathcal{D}} \Pi}_{\in A_{N+1}^1(N+1)}\right) \circ \left(\underbrace{1_X \otimes_{\mathcal{D}} \Pi}_{\in A_1^{N+1}(N+1)}\right) = \mu \underbrace{1_X}_{A_1^1} \in \operatorname{End}_{\mathcal{A}}(X).
\end{aligned}$$

As $\operatorname{End}_{\mathcal{A}}(X) = \operatorname{End}_{\mathcal{D}}(X) = \mathbb{C}$, this is just the number μ. We have just proven the existence of a monoidal algebra $\mathcal{A}$ with diagonal $\mathcal{D}$ and $\Pi(\mathcal{A}) = \Pi$, and with $\Theta(\mathcal{A}) = \mu$. □

As we observed in Proposition 4.7, the braiding c on $\mathcal{D}$ extends to a braiding on $\mathcal{A}$ if and only if $\Theta(\mathcal{A}) = 1$. If $\Theta \neq 1$ we use the braiding c' instead of c as in the previous proof, which does change Θ to 1. So the braiding c' can be extended to a braiding on $\mathcal{A}$ also in that case. It follows that all possible N extensions $\mathcal{A}$ of $\mathcal{D}$ can be given the structure of a braided category. We have shown

Corollary 4.9. *A fixed braiding of $\mathcal{D}$ extends to a braiding of only one of the N possible monoidal algebras of which it can be the diagonal. However, for a given other monoidal algebra $\mathcal{A}$ we can always find a braiding of $\mathcal{D}$ which does extend to a braiding of $\mathcal{A}$.*

5. Rigid Categories

We collect and (re)prove a number of basic results about rigidity in braided categories which are probably well-known to experts. This will be done in the context of ribbon tensor categories, so we need not worry about left- or right-rigidity.

Lemma 5.1. *Let $\mathcal{C}$ be a rigid semisimple ribbon tensor category. Then any simple object has nonzero dimension. In particular, the bilinear form $\langle a, b\rangle = \mathrm{tr}(a \circ b)$ on $\mathrm{End}(Z)$ is nondegenerate for any object Z in $\mathcal{C}$.*

Proof: Let X be a simple object in $\mathcal{C}$, with dual object Y. Let $i_X : \mathbb{1} \to X \otimes Y$, $d_X : Y \otimes X \to \mathbb{1}$, $i'_X : \mathbb{1} \to Y \otimes X$, and $d'_X : X \otimes Y \to \mathbb{1}$ be the corresponding left and right rigidity morphisms. As X is simple, the object $\mathbb{1}$ appears with multiplicity one in $X \otimes Y$. Let Π be the unique projection onto it. If $\dim X = 0$, then $(i_X \circ d'_X)^2 = 0$. Hence the morphism $i_X \circ d'_X$ is a nilpotent multiple of Π, and therefore it must be equal to 0. But this would contradict the rigidity axiom as follows:

$$0 = \left[\, 1_X \otimes d_X\right] \circ \left[(i_X \circ d'_X) \otimes 1_X\,\right] \circ \left[\, 1_X \otimes i'_X\right] =$$

$$= \left[(1_X \otimes d_X) \circ (i_X \otimes 1_X)\right] \circ \left[(d'_X \otimes 1_X) \circ (1_X \otimes i'_X)\right] = 1_X \circ 1_X = 1_X,$$

a contradiction (here the second equality follows from the rigidity axiom and from [18], Prop. XIV.3.5). $\square$

It will also be convenient to define partial trace operations, which are also known under the names contractions or conditional expectations. Let X and V be objects in $\mathcal{C}$. We define the map ε_V from $\mathrm{End}(V \otimes X)$ onto $\mathrm{End}(V)$ by

$$\varepsilon_V(b) = \frac{1}{\dim X}(1_V \otimes d'_X) \circ (b \otimes 1_Y) \circ (1_V \otimes i_X). \tag{5.1}$$

We have the following results:

Lemma 5.2. *Let $b \in \mathrm{End}(V \otimes X)$ and let $p = 1/(\dim X)\tilde{e}$ be the projection onto $\mathbb{1} \subset X \otimes Y$. Then*
- (a) *$\mathrm{tr}_{V\otimes X}(b) = \mathrm{tr}_V\left(\varepsilon_V(b)\right)$; in particular, if V is simple then $\varepsilon_V(b) = \mathrm{tr}_{V\otimes X}(b)1$.*
- (b) *$(1_V \otimes p) \circ (b \otimes 1_Y) \circ (1_V \otimes p) = \varepsilon_V(b) \otimes p$.*

Proof: These statements are easy consequences from the definitions (see also e.g. [31], Prop. 1.4). $\square$

We shall need the results of the last lemma in the following setting. Let $m \in \mathrm{End}\left(X^{\otimes 2}\right)$. Then we define the morphism $m_i \in \mathrm{End}\ (X^{\otimes k})$ by

$$m_i \ = \ 1_{i-1} \otimes m \otimes 1_{k-1-i},$$

where 1_r is the identity morphism of $X^{\otimes r}$. Then we have the following (see also e.g. [31], Prop. 1.4)

Corollary 5.3.
- (a) *(Markov property) If $a \in \mathrm{End}\left(X^{\otimes n}\right)$, then $\mathrm{tr}\left((a \otimes 1) \circ m_n\right) = \mathrm{tr}(a)\,\mathrm{tr}(m)$.*
- (b) *Assume that X is a self-dual object (see below) and that $X^{\otimes 2} \cong \oplus_{j=1}^d X_{\mu_j}$, and assume that we can write $1_X = \sum_j p_{\mu_j}$ as a sum of commuting projections $p_{\mu_j} \in \mathrm{End}\left(X^{\otimes 2}\right)$ such that $\mathrm{Im}(p_{\mu_j}) \cong X_{\mu_j}$. Then $p_2(p_{\mu_j} \otimes 1)p_2 = \frac{\dim X_{\mu_j}}{(\dim X)^2} p_2$.*

5.1. **Self-dual objects.** Let $\mathcal{C}$ be a semisimple ribbon tensor category, and let X be an object in $\mathcal{C}$ which is isomorphic to its dual. Hence we have $i = i_X : \mathbb{1} \to X^{\otimes 2}$, $d = d_X : X^{\otimes 2} \to \mathbb{1}$, $i' = i'_X : \mathbb{1} \to X^{\otimes 2}$, and $d' = d'_X : X^{\otimes 2} \to \mathbb{1}$ satisfying the left and right rigidity axioms. In the following we will denote the braiding morphism $c_{X,X} \in \mathrm{End}\left(X^{\otimes 2}\right)$ just by c, and $i \circ d'$ by $\tilde{e}$. The morphisms i_1 and i_2 are defined by

$$i_1 = i \otimes 1_1 : X \cong X \otimes \mathbb{1} \to X^{\otimes 3} \quad \text{and} \quad i_2 = 1_1 \otimes i : \mathbb{1} \otimes X \to X^{\otimes 3},$$

with d_1 and d_2 being morphisms from $X^{\otimes 3}$ to X defined similarly.

Lemma 5.4. *Let X be a simple self-dual object with dimension $\dim X$ and let $\tilde{r} \in F$ be the scalar such that $\theta_X = \tilde{r} 1_X$. Then there exists $\alpha \in \{\pm 1\}$ such that $c \circ i = \alpha \tilde{r}^{-1} i$, $\mathrm{tr}(c) = \tilde{r}/(\dim X)$ and $\mathrm{tr}(\tilde{e}) = 1/(\dim X)$ for the normalized categorical trace tr for $\mathrm{End}\left(X^{\otimes 2}\right)$.*

Proof: By definition, $\dim X = d' \circ i = \mathrm{Tr}(d' \circ i) = \mathrm{Tr}(i \circ d') = \mathrm{Tr}(\tilde{e})$, which implies the statement for $\mathrm{tr}(\tilde{e})$. As $\theta_{\mathbf{1}} = 1_{\mathbf{1}}$, it follows that

$$i = \theta_{X \otimes X} \circ i = c \circ c \circ (\theta_X \otimes \theta_X) \circ i = \tilde{r}^2 \ c \circ c \circ i.$$

As $c \circ i$ is a multiple of i, the first claim follows. This also implies that $i' = \alpha i$ and $d' = \alpha d$. Using the braiding axioms, we obtain the identity $c_1 \circ c_2 \circ i_1 = i_2$; after multiplying by d'_1 from the right, we obtain the equality $c_1 \circ c_2 \circ (i_1 \circ d'_1) = \alpha (i_2 \circ d'_2) \circ (i_1 \circ d'_1)$ in $\mathrm{End}\left(X^{\otimes 3}\right)$. Using the trace property and the Markov property, we obtain $\mathrm{tr}\left(c_1 \circ c_2 \circ (i_1 \circ d'_1)\right) = \alpha \tilde{r}^{-1}\, \mathrm{tr}(c)/(\dim X)$, which has to be equal to $\alpha \big(\mathrm{tr}(i \circ d')\big)^2 = \alpha/(\dim X)^2$. The claim follows from this. □

The following lemma corrects the statement of Lemma 3.2 in [35]; the proof there would have been sufficient for the purposes in that paper and also for this paper.

Lemma 5.5. *The algebra generated by $\mathrm{End}\left(X^{\otimes 2}\right) \otimes 1$ and by $\tilde{e}_2$ acts irreducibly on the space $\mathrm{Hom}\left(X, X^{\otimes 3}\right)$, via concatenation.*

Proof: We use the notations as in Corollary 5.3,(b), with $p_{\mu,1} = p_\mu \otimes 1$. Observe that

$$(p_{\mu,1} \circ \tilde{e}_2 \circ p_{\nu,1}) \circ (p_{\kappa,1} \circ \tilde{e}_2 \circ p_{\gamma,1}) = \delta_{\nu,\kappa} (\dim X)\, \mathrm{tr}(p_\nu) (p_{\mu,1} \circ \tilde{e}_2 \circ p_{\gamma,1}).$$

Hence the set $\{p_{\mu_i,1} \circ \tilde{e}_2 \circ p_{\mu_j,1},\ i,j = 1,2,\ \dots\ d\}$ spans a full $d \times d$ matrix algebra. It obviously does not act trivially on $\mathrm{Hom}\left(X, X^{\otimes 3}\right)$. As $\dim \mathrm{Hom}\left(X, X^{\otimes 3}\right) = \dim \mathrm{End}\left(X^{\otimes 2}\right) = d$, the claim follows. □

6. Categories of orthogonal or symplectic type

6.1. **Combinatorial data.** We fix some notations for the representation category of a full orthogonal group $O(N)$ or a symplectic group $Sp(N)$. For these groups the defining or vector representations have dimension N (in the orthogonal case) and dimension $2N$ (in the symplectic case) respectively.

It is well-known that the isomorphism classes of simple representations of $O(N)$ are labeled by Young diagrams with at most N boxes in the first two columns; simple representations of $Sp(N)$ are labeled by Young diagrams with at most N rows. We call such Young diagrams permissible (for the respective group).

It is easy to describe the decomposition of the tensor product of a simple representation with the vector representation. Let X_λ be a simple object in $\mathcal{C}$ corresponding to the Young diagram λ, and let $X = X_{[1]}$ be the object corresponding to the vector representation (which is labeled by the Young diagram with one box). Then $X_\lambda \otimes X$ is the direct sum of simple representations labeled by all permissible Young diagrams μ which are obtained from λ by removing or adding a box from/to λ. While tensoring with the vector representation would not per se describe the Grothendieck semiring, it is all that we need for our purposes together with the braiding (see Prop. 8.6).

In the following, we denote by $[1^n]$ and by $[n]$ the Young diagrams with all its n boxes in its first column and in its first row respectively. The simple object $X_{[1^n]}$ corresponds to the full antisymmetrization of the n-th tensor power $X^{\otimes n}$ of the vector representation of the orthogonal group. In the representation category of symplectic groups it would correspond to the unique simple subrepresentation in the n-th antisymmetrization of the vector representation which has not already appeared in the smaller tensor powers. We obtain as a special case of the tensor product rule described above

$$X_{[1^m]} \otimes X \;\cong\; X_{[1^{m+1}]} \oplus X_{[2,1^{m-1}]} \oplus X_{[1^{m-1}]}, \quad 1 \le m < N; \tag{6.1}$$

if $m = N$, the right hand side above would be isomorphic to $X_{[1^{N-1}]}$ in the orthogonal case, and to $X_{[2,1^{N-1}]} \oplus X_{[1^{N-1}]}$ in the symplectic case.

6.2. **Fusion categories.** There also exist braided tensor categories whose Grothendieck semirings are quotients of the ones described in the last subsection. In these cases, we can describe the labeling set for its simple objects by also applying analogous restrictions to the rows of Young diagrams as we had before for columns. We have the following three cases, for fixed $N, M \in \mathbb{N}$:

(a) orthogonal fusion category: the simple objects are labeled by Young diagrams with $\le N$ boxes in its first two columns and with $\le M$ boxes in its first two rows,
(b) ortho-symplectic fusion category: the simple objects are labeled by Young diagrams with $\le N$ boxes in its first two columns and with $\le M$ boxes in its first row (i.e. with $\le M$ columns),
(c) symplectic fusion category: the simple objects are labeled by Young diagrams with at most N boxes in the first column and with at most M boxes in the first row.

Tensoring with the object labeled by the Young diagram with one box (the analog of the 'vector representation' in this context) is as before, with now only those objects allowed at the right hand side which satisfy the conditions for the labeling set of simple objects in the corresponding fusion category. In particular, this simple tensor product rule allows to compute the multiplicity of an object X_λ in $X^{\otimes n} = X_{[1]}^{\otimes n}$ by induction.

6.3. **Definition and examples.** In the rest of the paper, we have the following assumptions: All categories are supposed to be rigid, strictly monoidal, semisimple, braided $\mathbb{C}$-categories. We say that such a category, say $\mathcal{C}$, is of *orthogonal* or *symplectic type* if its Grothendieck semiring is the one of a representation category of $O(N)$ (including $O(\infty)$) or $Sp(N)$, or of one of the associated fusion categories, as described in the last two subsections. Here are examples for such categories:

a) By definition, the representation categories $\mathrm{Rep}\left(O(N)\right)$ and $\mathrm{Rep}\left((Sp(N)\right)$ are tensor categories of orthogonal resp. symplectic type, which have symmetric braiding.

b) It is well-known that the representation category of the Drinfeld-Jimbo quantum group $U_q\mathfrak{g}$ associated to the semisimple Lie algebra $\mathfrak{g}$ is semisimple and that $\mathrm{Rep}(U_q\mathfrak{g})$ has the same Grothendieck semiring as $\mathrm{Rep}(\mathfrak{g})$, if q is not a root of unity. As $\mathrm{Rep}(\mathfrak{sp}_N)$ is equivalent to $\mathrm{Rep}\left(Sp(N)\right)$, $\mathrm{Rep}(U_q\mathfrak{sp}_N)$ is a braided tensor category of symplectic type. It is also possible to construct braided tensor categories of orthogonal type as a semidirect product of a subcategory of $\mathrm{Rep}(U_q\mathfrak{so}_N)$ with $\mathrm{Rep}(\mathbb{Z}/2)$.

c) If q is a root of unity, H.H. Andersen defined the category of tilting modules of $U_q\mathfrak{g}$. This category contains as a quotient a semisimple category with only finitely many equivalence classes of simple objects. These are examples of fusion categories. One can construct the fusion categories of the last section from these quotient categories in complete analogy to the construction sketched in (b).

d) The existence of fusion categories was suggested by physicists in conformal field theory. In particular, this implied the existence of a highly nontrivial tensor product for representations of affine Kac-Moody algebras resp. loop groups. A mathematically rigorous definition was given by

Kazhdan and Lusztig in the Kac-Moody case (see [20], [21],[22]) and by Wassermann in [40] for loop groups. The equivalence between these categories and the ones defined by Andersen was shown by Finkelberg [12].

e) It is also possible to construct orthogonal and symplectic categories by topological methods as quotients of the tangle category (see [36]). This approach is closest to the set-up in this paper. It will be described in more detail in Section 9.2. A similar approach also works for Lie type A (see [7]).

6.4. **Low tensor powers.** As $\mathbb{1}$ is a subobject of $X^{\otimes 2}$, any simple subobject of $X^{\otimes(n-2)}$ is also isomorphic to a simple subobject of $X^{\otimes n}$. Hence we can write $X^{\otimes n}$ as a direct sum $X_{(n-2)} \oplus X_n$, where $X_{(n-2)}$ is a direct sum of simple objects each of which is isomorphic to a subobject of $X^{\otimes(n-2)}$ and X_n is a direct sum of simple objects which are not isomorphic to any subobject of $X^{\otimes(n-2)}$. By semisimplicity of $\mathcal{C}$, we get from this the decomposition

$$\text{End}\left(X^{\otimes n}\right) \cong \text{End}\left(X_{(n-2)}\right) \oplus \text{End}\left(X_n\right). \tag{6.2}$$

Lemma 6.1. *The set $\tilde{\mathcal{B}} = \{1, c, \tilde{e}\} \subset \text{End}\left(X^{\otimes 2}\right)$ is linearly independent. In particular, c acts via different scalars on $X_{[2]}$ and on $X_{[1^2]}$.*

Proof: Assume that $\tilde{\mathcal{B}}$ is not linearly independent. Then we can assume $c = \alpha 1 + \beta \tilde{e}$, with $\alpha, \beta \in F$, as otherwise the noninvertible $\tilde{e}$ would be proportional to 1. But then all the c_i's just act as scalars in End (X_n). Let now f resp. $\tilde{f}$ be the projections onto the simple subobjects $X_{[1^2]}$ resp $X_{[2]}$ of $X^{\otimes 2}$. Then we get, using $n = 4$ and the braiding with $c_{(2)} = c_{X^{\otimes 2}, X^{\otimes 2}} = c_2 c_1 c_3 c_2$

$$f \otimes \tilde{f} = f_1 c_{(2)} \tilde{f}_1 c_{(2)}^{-1} \quad \in \text{End}\left(X_{(n-2)}\right),$$

where the last inclusion follows from the fact that $f_1 \tilde{f}_1 = 0$ and that $c_{(2)}$ only acts as scalar in $\text{End}(X_n)$, i.e. conjugation by it induces the trivial automorphism in $\text{End}(X_n)$.

As $\text{End}\left(X_{[1^2]} \otimes X_{[2]}\right) \cong f_1 \tilde{f}_3 \text{ End}\left(X^{\otimes 4}\right) f_1 \tilde{f}_3 \subset \text{End}\left(X_{(2)}\right)$, we obtain that $X_{[1^2]} \otimes X_{[2]}$ decomposes into a direct sum of simple modules which already appear in $X^{\otimes 2}$ (i.e. they are isomorphic to $\mathbb{1}, X_{[1^2]}$ or $X_{[2]}$); this contradicts the tensor product rules for orthogonal and symplectic groups. $\square$

Lemma 6.2. *The space $\text{Hom}\left(X, X^{\otimes 3}\right)$ has the basis $\mathcal{B} = \{i_2, c_1 \circ i_2, \tilde{e}_1 \circ i_2 = i_1'\}$.*

Proof: This is a special case of Frobenius reciprocity: the map $a \in \text{End}\left(X^{\otimes 2}\right) \mapsto (a \otimes 1) \circ i_2$ has the inverse map $b \in \text{Hom}\left(X, X^{\otimes 3}\right) \mapsto (1_2 \otimes d) \circ (b \otimes 1)$. The claim now follows from Lemma 6.1. $\square$

6.5. **Matrix representations.** We define the quantity $d(X)$ by $d(X) = d \circ i$. Recall from the last section that $d(X) = \alpha \dim(X)$ (see Lemma 5.4).

Lemma 6.3. *There are scalars r, q and a fourth root of unity γ such that*

(a) *the element $t = \gamma c$ has the eigenvalues q, $-q^{-1}$ and r^{-1} for the submodules $X_{[2]}$, $X_{[1^2]}$ and $\mathbb{1}$ of $X^{\otimes 2}$ respectively,*
(b) *if $q \neq q^{-1}$, then*

$$d(X) = \frac{r - r^{-1}}{q - q^{-1}} + 1 = \frac{q^{-1}(r+q)(q - r^{-1})}{q - q^{-1}}. \tag{6.3}$$

(c) $\text{tr}(t) = r/d(X)$.

Proof: It will be useful to compute matrix representations of the elements c_i and $\tilde{e}_i$, $i = 1, 2$, acting on $\text{Hom}(X, X^{\otimes 3})$ via concatenation. We will use the basis $\{i_2, c_1 \circ i_2, i_1\}$. We claim that if the eigenvalues of c are λ_1, λ_2 and λ_3, then we obtain the matrix representations

$$(6.4)\qquad c_1 \mapsto \begin{pmatrix} 0 & -\lambda_1\lambda_2 & 0 \\ 1 & \lambda_1+\lambda_2 & 0 \\ 0 & \lambda_3(\lambda_1^{-1}+\lambda_2^{-1}) & \lambda_3 \end{pmatrix} \quad \text{and} \quad c_2 \mapsto \begin{pmatrix} \lambda_3 & 0 & \lambda_1\lambda_2(\lambda_1+\lambda_2) \\ 0 & 0 & -\lambda_1\lambda_2 \\ 0 & 1 & \lambda_1+\lambda_2 \end{pmatrix}.$$

To see this observe that we have the obvious relations $c_j \circ i_j = \lambda_3 i_j$ for $j = 1, 2$, and, from the braiding axiom, $c_2 \circ (c_1 \circ i_2) = i_1$. This determines two of the three columns of c_j, $j = 1, 2$. Of the remaining column, two entries are computed using the fact that the matrix must have determinant $\lambda_1\lambda_2\lambda_3$ and trace $\lambda_1+\lambda_2+\lambda_3$. The remaining entries can be computed checking the braid relation $c_1 \circ c_2 \circ c_1 = c_2 \circ c_1 \circ c_2$. Moreover, using the braiding relation, we get $c_1 \circ c_2 \circ i_1 = i_2$, while the corresponding matrices, applied to i_1 would give $(\lambda_1\lambda_2)^2 i_2$. Hence we also have $(\lambda_1\lambda_2)^2 = 1$, and we can assume $\lambda_1 = \gamma^{-1}q$, $\lambda_2 = -\gamma^{-1}q^{-1}$ and $\lambda_3 = \gamma^{-1}r^{-1}$ for certain complex numbers r and q and for γ a fourth root of unity.

Now it easily follows and the results of Lemma 5.4,

$$(6.5)\qquad \tilde{e}_1 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha & \alpha\lambda_3 & \dim X \end{pmatrix} \quad \text{and} \quad \tilde{e}_2 \mapsto \begin{pmatrix} \dim X & \alpha\lambda_3^{-1} & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Comparing the $(3, 2)$-matrix entries in the equality $\tilde{e}_1 c_1 = \lambda_3 \tilde{e}_1$, we obtain

$$\big(\lambda_3(\lambda_1^{-1}+\lambda_2^{-1})\big)\big(\dim X\big) = \alpha\big(\lambda_3^2 + \lambda_1\lambda_2 - \lambda_3(\lambda_1+\lambda_2)\big).$$

If $\lambda_1 + \lambda_2 \neq 0$, this gives the formula for the dimension and for $d(X)$ as stated, after substituting r and q into the eigenvalues as above. It follows from this and Lemma 5.4, with $\tilde{r} = \alpha\lambda_3^{-1}$ that $\mathrm{tr}(t) = \mathrm{tr}(\gamma c) = r/d(X)$, as stated.

If $\lambda_1 = -\lambda_2$, we deduce from the last equation that $\lambda_3^2 = -\lambda_1\lambda_2 = \pm 1 = \lambda_2^2 = \lambda_1^2$. This implies that two of the three eigenvalues of c are identical and that the eigenvalues of t are contained in the set $\{\pm 1\}$. □

Lemma 6.4. *Let t be as in Lemma 6.3. If t has less than three distinct eigenvalues, then necessarily its eigenvalues are ± 1.*

Proof: We can rule out $\lambda_1 = \lambda_2$ by Lemma 6.1. Assume now that $\lambda_1 = \lambda_3$ or $\lambda_2 = \lambda_3$, which would imply $r = -q$ or $r = q^{-1}$ for the eigenvalues of t. If $\lambda_1 + \lambda_2 \neq 0$, we obtain $\dim X = 0$ from the computations of the previous lemma, which would contradict rigidity. If $\lambda_1 + \lambda_2 = 0$, the claim follows from the last paragraph of the proof of the last lemma. □

6.6. **Relations.** We can now summarize the results of this section as follows: Let $e = i \circ d = \alpha\tilde{e}$.

Proposition 6.5.

(a) *Assume that c has three distinct eigenvalues, and let $t = \gamma c$ be as in Lemma 6.3. Then we can define the element $e \in \mathrm{End}\left(X^{\otimes 2}\right)$ also by $t - t^{-1} = (q - q^{-1})(1 - e)$. We then have the relations*
   - (R1) *$t_i e_i = r^{-1} e_i$, for $i = 1, 2, \ldots n-1$, and*
   - (R2) *$e_i t_{i-1}^{\pm 1} e_i = r^{\pm 1} e_i$, for $i =, 2, \ldots n-1$.*

(b) *If c has fewer than three eigenvalues, then the representation of the braid group B_n given by the morphisms t_i factors through the symmetric group S_n. Moreover, the elements t_i and e_i, $i = 1, 2 \ldots n-1$ generate a quotient of Brauer's centralizer algebra.*

(c) *We also have $\mathrm{tr}(t) = r/d(X)$ and $\mathrm{tr}(e) = 1/d(X)$ and $\mathrm{tr}\left((a \otimes 1)\chi_{n-1}\right) = \mathrm{tr}(a)\,\mathrm{tr}(\chi)$ for $\chi \in \{t, e\}$ in both cases; here tr is the normalized trace on $\mathrm{End}\left(X^{\otimes}\right)$ and $a \in \mathrm{End}\left(X^{\otimes n-1}\right)$.*

Proof: By definition, e is a multiple of the eigenprojection of t for the eigenvalue r^{-1}. It can be seen e.g. from the explicit matrix representations, see 6.5, that this multiple is $d(X)$. The alternative formula for e can now be checked easily, as well as (R1). Part (c) follows from Lemma 6.3 resp

Lemma 5.4 for the values of $\mathrm{tr}(t)$ and $\mathrm{tr}(e)$, and from Corollary 5.3 for the Markov property. Using the relation between $t^{\pm 1}$ and e in part (a) of the statement, one also computes $\mathrm{tr}(t^{-1}) = r^{-1}/d(X)$.

By functoriality, it suffices to check Relation (R2) for $i = 2$. This follows from Lemma 5.2(b) and (a), and from the values of $\mathrm{tr}(t^{\pm 1})$ which have already been computed. The proof for part (b) will be given in Section 7.4. □

7. q-Deformation of Brauer's centralizer algebra

After having determined properties of braiding morphisms for braided tensor categories of orthogonal or symplectic types, we now go the opposite way. We use the relations obtained in the last section to define abstract algebras which turn out to be Brauer's centralizer algebras (see [8]) and a q-deformation of it which was discovered in connection with Kauffman's link invariant (see [6] and [30]; here we follow the presentation in [43], p 399/400).

7.1. **Hecke algebras.** We first need a simpler class of algebras. The Iwahori-Hecke algebra $\mathcal{H}_n(q^2)$ of type A_{n-1} is the algebra defined over the field F by generators $\tilde{T}_i$, $i = 1, 2, \ \ldots\ n-1$, which satisfy the braid relations and the quadratic relation $\tilde{T}_i^2 = (q - q^{-1})\tilde{T}_i + 1$; here q is a fixed element in F. We have the following well-known theorem:

Theorem 7.1. *If q^2 is not a root of unity of order $\leq n$, then $\mathcal{H}_n(q^2)$ is isomorphic to the group algebra FS_n of the symmetric group S_n.*

One of the consequences of the last theorem is that the irreducible representations of $\mathcal{H}_n(q^2)$ are labeled by Young diagrams with n boxes if $\mathcal{H}_n(q^2)$ is semisimple. In that case, let $\tilde{P}_{[1^n]}$ be the central idempotent corresponding to the one-dimensional representation $\tilde{T} \mapsto -q^{-1}$. Let $A \otimes 1$ denote the element in $\mathcal{H}_{n+1}$ obtained from the element $A \in \mathcal{H}_n$ under the natural embedding of $\mathcal{H}_n$ into $\mathcal{H}_{n+1}$. It is well-known that we have

$$\tilde{P}_{[1^n]} \otimes 1 = \tilde{P}_{[1^{n+1}]} + \tilde{P}_{[2,1^{n-1}]}, \tag{7.1}$$

where $\tilde{P}_{[2,1^{n-1}]}$ is an idempotent in the simple component of $\mathcal{H}_{n+1}$ labeled by the Young diagram $[2, 1^{n-1}]$. Let $[n]_q = (q^n - q^{-n})/(q - q^{-1})$.

Lemma 7.2. *We have the following identities in $\mathcal{H}_n$, for $m = 1, 2, \ \ldots\ n-1$:*

(a) $\tilde{P}_{[1^{m+1}]} = \frac{1}{[m+1]_q}\left(q^m \tilde{P}_{[1^m]} - [m]_q \tilde{P}_{[1^m]}\tilde{T}_m\tilde{P}_{[1^m]}\right)$

(b) $\tilde{P}_{[1^m]}\tilde{T}_m\tilde{P}_{[2,1^{m-2}]}\tilde{T}_m\tilde{P}_{[1^m]} = \frac{[m-1]_q[m+1]_q}{[m]_q^2}\left(\tilde{P}_{[1^m]} - \tilde{P}_{[1^{m+1}]}\right) = \frac{[m-1]_q}{[m]_q}\tilde{P}_{[1^m]}\left(\tilde{T}_m + q^{-1}1\right)\tilde{P}_{[1^m]}$.

Proof: These identities follow as special cases from properties of path idempotents connected to Hoefsmit's orthogonal representations of Hecke algebras (see e.g. [41], Cor 2.3). They can also be proved by induction on m as follows:

We can write $\tilde{T}_i = (q + q^{-1})\tilde{E}_i - q^{-1}1$, where $\tilde{E}_i$ is the eigenprojection for the eigenvalue q of $\tilde{T}_i$. Then one shows by induction on m, using $\tilde{P}_{[1^m]}\tilde{E}_{m-1} = 0$ and $\tilde{E}_i\tilde{E}_{i-1}\tilde{E}_i = \tilde{E}_{i-1}\tilde{E}_i\tilde{E}_{i-1} - \frac{[1]_q}{[2]_q^2}(\tilde{E}_i - \tilde{E}_{i-1})$ that

$$\tilde{P}_{[1^m]}\tilde{E}_m\tilde{P}_{[1^m]}\tilde{E}_m = \frac{[m+1]_q}{[m]_q[2]_q}\tilde{P}_{[1^m]}\tilde{E}_m$$

and

$$\tilde{P}_{[1^{m+1}]} = \tilde{P}_{[1^m]} - \frac{[m]_q[2]_q}{[m+1]_q}\tilde{P}_{[1^m]}\tilde{E}_m\tilde{P}_{[1^m]}.$$

Claim (a) follows from the second equation. Claim (b) follows by substituting $\tilde{P}_{[2,1^{m-1}]} = \tilde{P}_{[1^{m+1}]} - \tilde{P}_{[1^m]}$, and then applying (a) for $\tilde{P}_{[1^{m+1}]}$. □

7.2. **Definitions.** The algebra $\mathcal{D}_n(r,q)$, depending on two parameters r and q, is given by generators $T_1,\ T_2\ ...\ T_{n-1}$, which satisfy the braid relations and

(R1) $E_iT_i = r^{-1}E_i$,

(R2) $E_iT_{i-1}^{\pm 1}E_i = r^{\pm 1}E_i$,

where E_i is defined by the equation

(D) $(q-q^{-1})(1-E_i) = T_i - T_i^{-1}$.

Remarks : It is easy to read off from the defining relations the following facts:

(a) Let $\mathcal{I}_n$ be the ideal of $\mathcal{D}_n$ generated by E_{n-1}. Then $\mathcal{D}_n/\mathcal{I}_n \cong \mathcal{H}_n(q^2)$, with the isomorphism given by $T_i \mapsto \tilde{T}_i$.

(b) $\mathcal{I}_n \cong \mathcal{D}_{n-1} \otimes_{\mathcal{D}_{n-2}} \mathcal{D}_{n-1}$ as a $\mathcal{D}_{n-1} - \mathcal{D}_{n-1}$ bimodule, where the isomorphism is given by $D_1 \otimes D_2 \mapsto D_1E_{n-1}D_2$ for $D_1, D_2 \in \mathcal{D}_{n-1}$.

(c) If $\mathcal{D}_n$ is semisimple, $\mathcal{D}_n \cong \mathcal{I}_n \oplus \mathcal{H}_n$.

(d) The T_i's satisfy the cubic equation $(T_i - r^{-1})(T_i + q^{-1})(T_i - q) = 0$.

7.3. **Structure of q-Brauer algebras.** The following Theorem determines the structure of $\mathcal{D}_n(r,q)$ if it is semisimple (see [6], Theorem 3.7 and [43], Theorem 3.5 and Cor 5.6):

Theorem 7.3.

(a) *The algebra $\mathcal{D}_n(r,q)$ is semisimple for generic values of r and q (see Theorem 7.4 for more specific information). In this case, it has dimension $1\cdot 3\cdot 5\ ...\ (2n-1)$ and its simple components are labeled by the Young diagrams with n, $n-2$, $n-4$, ..., 1 resp. 0 boxes.*

(b) *The decomposition of a simple $\mathcal{D}_{n,\lambda}$ module $V_{n,\lambda}$ into simple $\mathcal{D}_{n-1}$ modules is given by*

$$V_{n,\lambda} \cong \bigoplus_{\mu} V_{n-1,\mu}, \tag{7.2}$$

where the summation goes over all Young diagrams μ which can be obtained by either taking away or, if λ has less than n boxes, by adding a box to λ. The labeling of simple components is uniquely determined by this restriction rule and the convention that the eigenprojection of T_1 corresponding to its eigenvalue q is labeled by the Young diagram [2].

(c) *For diagrams λ with n boxes, $V_{n,\lambda}$ becomes an $\mathcal{H}_n(q^2)$ module via the homomorphism of Remark (a) in Section 7.2.*

(d) $\mathcal{D}_{n+1} = \mathrm{span}\left\{\ A\chi B \mid A, B \in \mathcal{D}_n,\ \chi \in \{1, T_n, E_n\}\right\}$.

We leave it to the reader to check, using the inductive rule in Theorem 7.3, (b) (see also [6], Fig. 8) that $\mathcal{D}_1(r,q) \cong F$, $\mathcal{D}_2(r,q) \cong F^3$ and, with $M_k(F)$ denoting the algebra of all $k\times k$ matrices,

$$\mathcal{D}_3(r,q) \cong M_3(F) \oplus F \oplus M_2(F) \oplus F.$$

It is an easy exercise to show (using relations (1)-(10) in [43], p. 400) that the 3-dimensional simple component contains a minimal left ideal spanned by $\{E_2, T_1E_2, E_1E_2\}$, and that the matrices which describe the action of the elements T_i and E_i, $i = 1,2$ with respect to this basis coincide with the ones in Eq 6.4 and 6.5.

7.4. **Brauer algebras.** Brauer defined abstract finite dimensional algebras $\mathcal{B}D_n = \mathcal{B}D_n(x)$ (see [8]) depending on a parameter x. These abstract algebras are easiest described by graphs. We will not give this description here (see [8]).

The following description will suffice for our purposes: The algebras $\mathcal{B}D_n$ can be defined via generators T_i' and E_i', $i = 1,2,\ ...\ n-1$. For $x = N > n$, we obtain a faithful surjective representation of $\mathcal{B}D_n(N)$ onto $\mathrm{End}_{O(N)}\left(V^{\otimes n}\right)$ which maps T_i' to the permutation of the i-th and $(i+1)$-st factor, and it maps E_i' to the element $\tilde{e}_i$ defined for this category as in Section 5.1; here V is the N-dimensional vector representation of $O(N)$. Similarly, one obtains a surjective map $\mathcal{B}D_n(-2N)$ onto $\mathrm{End}_{Sp(N)}\left(V^{\otimes n}\right)$.

The commutation relations between the elements T_i' and E_i' are exactly the same ones as for the elements T_i and E_i in $\mathcal{D}_n$. In particular, the elements T_i', E_i' commute with T_j' and E_j' whenever $|i-j|\geq 2$. In fact, the relations for $x=N$ follow from the ones in $\mathcal{D}_n(q^{N-1},q)$ in the limit for $q\to 1$. (see e.g. [6], Section 5 or [43], p 401 for details).

Conclusion of the proof of Prop.6.5: Evaluating the matrices in the proof of Lemma 6.3 for $r=q=1$, we obtain matrices for t_i, e_i, $i=1,2$ which only depend on $d(X)$. In particular $t_i^2=1$ for $i=1,2$. Moreover, if $d(X)=N$, these matrices have to satisfy the same relations as the corresponding elements in Rep $\big(O(N)\big)$. By functoriality, the elements $t_i, e_i, t_{i+1}, e_{i+1}$ satisfy the same relations as the elements t_1, e_1, t_2, e_2, and generators whose indices differ by at least 2 commute. Hence the elements t_i, e_i generate an algebra isomorphic to a quotient of Brauer's centralizer algebra. $\square$

7.5. **q-Dimensions.** We also need a general formula for q-dimensions of orthogonal and symplectic groups. Let $[y+n]_q=(rq^n-r^{-1}q^{-n})/(q-q^{-1})$. Then we define for each Young diagram λ the rational function

$$Q_\lambda(r,q) \;=\; \prod_{(j,j)\in\lambda}\frac{(r-q^{-2\lambda_j+2j-1})(r^{-1}+q^{-2\lambda_j'+2j-1})}{1-q^{-2h(j,j)}}\quad \prod_{\substack{(i,j)\in\lambda\\ i\neq j}}\frac{[y+d(i,j)]_q}{[h(i,j)]_q}; \tag{7.3}$$

here (i,j) denotes the box in the i-th row and j-th column of λ, λ_i (λ_j') is the number of boxes in the i-th row (j-th column) of λ. Moreover, the quantity $d(i,j)$ and the hook length $h(i,j)$ are defined by

$$d(i,j) \;=\; \begin{cases}\lambda_i+\lambda_j-i-j+1 & \text{if } i\leq j,\\ -\lambda_i'-\lambda_j'+i+j-1 & \text{if } i>j.\end{cases} \tag{7.4}$$

and

$$h(i,j)=\lambda_i-i+\lambda_j'-j+1 \tag{7.5}$$

We will need these functions primarily for the special case of a Young diagram $[1^m]$ whose only column contains exactly m boxes. In this case, we obtain

$$Q_{[1^m]}(r,q)=\frac{(r-q^{-1})(r^{-1}+q^{1-2m})}{1-q^{-2m}}\prod_{j=1}^{m-1}\frac{rq^{1-j}-r^{-1}q^{j-1}}{q^j-q^{-j}}. \tag{7.6}$$

One checks similarly that for the Young diagram $[2,1^{m-2}]$ with two boxes in the first row and one box in the next $m-2$ rows one obtain

$$Q_{[2,1^{m-2}]}(r,q)=\frac{(r-q^{-3})(r^{-1}+q^{3-2m})}{1-q^{-2m}}\,\frac{[y+1]_q[y+2-m]_q}{[1]_q[m-2]_q}\prod_{j=1}^{m-3}\frac{rq^{1-j}-r^{-1}q^{j-1}}{q^j-q^{-j}}. \tag{7.7}$$

The rational functions $Q_\lambda(r,q)$ have obvious analogs $\hat{Q}_\lambda(y)$ for the Brauer algebras. They are essentially defined by replacing q-numbers in the definition of Q_λ by ordinary numbers. More precisely, we have

$$\hat{Q}_\lambda(y)=\prod_{(i,j)\in\lambda}\frac{y+d(i,j)}{h(i,j)}, \tag{7.8}$$

7.6. **Markov traces and semisimplicity.** The algebras $\mathcal{D}_n(r,q)$ carry an important trace functional which we will describe in two different ways. The existence of the trace was originally derived from the existence of Kauffman's link invariant (see [6],[30]). The equivalent description in the semisimple case follows from [43], Theorem 3.6 and Theorem 5.5. A more algebraic existence proof can be given using the theory of quantum groups (see e.g. [43] and [31], Lemma 3.1).

Theorem 7.4.

(a1) *There exists a trace functional* $\mathrm{tr}_{\mathcal{D}}$ *on* $\mathcal{D}_n(r,q)$ *which is uniquely determined inductively by* $\mathrm{tr}_{\mathcal{D}}(1) = 1$, $\mathrm{tr}_{\mathcal{D}}(T_i) = r/d(X)$, $\mathrm{tr}_{\mathcal{D}}(E_i) = 1/d(X)$ *and by* $\mathrm{tr}_{\mathcal{D}}(A\chi B) = \mathrm{tr}_{\mathcal{D}}(AB)\,\mathrm{tr}_{\mathcal{D}}(\chi)$, *where* $A, B \in D_{n-1}$ *and* $\chi \in \{T_{n-1}, E_{n-1}, 1\}$*; here* $d(X) = (r - r^{-1})/(q - q^{-1}) + 1$ *is defined as in Lemma 6.3,(b).*

(a2) *If* $\mathcal{D}_n$ *is semisimple, the functional* $\mathrm{tr}_{\mathcal{D}}$ *in (a) is completely determined by* $\mathrm{tr}_{\mathcal{D}}(p) = Q_\lambda/d(X)^n$, *where* p *is a minimal idempotent in* $\mathcal{D}_{n,\lambda}$.

(b) *Conversely, if* q^2 *is not a primitive* l*-th root of unity for* $1 < l \le n$, *and if* $Q_\lambda(r,q) \neq 0$ *for all Young diagrams* λ *with less than* n *boxes, then the algebra* $\mathcal{D}_n(r,q)$ *is semisimple.*

(c) *If* $r = q^{N-1}$, $Q_\lambda(q^{N-1}, q)$ *is equal to the* q*-dimension of the highest weight module* V_λ *of* $O(N)$. *If* $r = -q^{2N-1}$, $(-1)^{|\lambda|}\, Q_\lambda(-q^{2N+1}, q)$ *is equal to the* q*-dimension of the highest weight module* V_λ *of* $Sp(N)$, *where* $|\lambda|$ *is the number of boxes of* λ. *The* q*-dimension of* V_λ *is defined to be the character of the element* $q^{2\rho}$, *acting on* V_λ, *where* ρ *is half the sum of the positive weights of the corresponding semisimple Lie algebra.*

(d) *One can similarly define the Markov trace for the Brauer algebras* $\mathcal{B}D_n(d(X))$, *where now the functions* $Q_\lambda(r,q)$ *are replaced by the polynomials* $\hat{Q}_\lambda(d(X))$ *(with* $r = q = 1$*).*

7.7. **Quotients of $\mathcal{D}_n(r,q)$.** It will be important to compute the quotient of $\mathcal{D}_n(r,q)$ modulo the annihilator ideal $\mathcal{A}_n$ of $\mathrm{tr}_{\mathcal{D}}$, i.e. $\mathcal{A}_n = \{A \in \mathcal{D}_n, \mathrm{tr}_{\mathcal{D}}(AB) = 0 \text{ for all } B \in \mathcal{D}_n\}$. In the following we assume q^2 to be a primitive l-th root of unity (with $l = \infty$ covering the case $q^2 = 1$ or q not a root of unity).

(a1) If $r = q^{N-1}$ or if $r = -q^{N-1}$ for N odd, with q^2 not a root of unity, then $\mathcal{D}_n/\mathcal{A}_n \cong \mathrm{End}_{O(N)}\left(V^{\otimes n}\right)$, where V is the vector representation of the orthogonal group $O(N)$. If $r = -q^{2N+1}$ with q^2 not a root if unity, then $\mathcal{D}_n/\mathcal{A}_n \cong \mathrm{End}_{Sp(N)}\left(V^{\otimes n}\right)$, where V is the vector representation of the symplectic group $Sp(N)$.

(a2) If r is equal to $\pm$ a negative power of q and q^2 is not a root of unity, then again $\mathcal{D}_n/\mathcal{A}_n$ is isomorphic to $\mathrm{End}_G\left(V^{\otimes n}\right)$, with V the vector representation of an orthogonal or symplectic group G. The group can be determined from $(a1)$ after replacing r by $-r^{-1}$. The results listed in (a1) and (a2) are proved in [43], Corollary 5.6.

(b) If q^2 is a primitive l-th root of unity, we can find positive integers $n, m < l$ such that $r = \pm q^n$ and $r = \pm q^{-m}$ (where the signs may or may not match). Then we can find restrictions for the number of boxes in the first (two) row(s) as well as in the first (two) column(s), as it was described in parts (a1) and (a2). Then again $\mathcal{D}_n/\mathcal{A}_n$ is isomorphic to $\mathrm{End}\left(X^{\otimes n}\right)$, where now X is the 'vector representation' of the corresponding fusion category, as described in Section 6.2. See [43], Theorem 6.4 for a somewhat more explicit description and a proof.

7.8. **Reparametrization.** It is easy to see that for the category $\mathcal{C}$ generated by X, we have several different braiding structures. Indeed, it is easy to check that replacing $c = c_{X,X}$ by $-c$, c^{-1} or $-c^{-1}$ again gives a braiding structure. Moreover, we have made a choice by labeling the object corresponding to the eigenvalue q by the Young diagram [2], and not by $[1^2]$. These observations are reflected on the level of the algebras $\mathcal{D}_n(r,q)$ as follows:

(a) There exist algebra isomorphisms $\mathcal{D}_n(r,q) \cong \mathcal{D}_n(-r,-q) \cong \mathcal{D}_n(-r^{-1}, -q^{-1}) \cong$
$\cong \mathcal{D}_n(r^{-1}, q^{-1})$ given by

$$T_i \mapsto -T_i(-r,-q) \mapsto -T_i^{-1}(-r^{-1}, -q^{-1}) \mapsto T_i^{-1}(r^{-1}, q^{-1}),$$

where $T_i(r',q')$, $i=1,2,\ \dots\ n-1$ are the generators of the algebra $\mathcal{D}_n(r',q')$. These isomorphisms preserve the labeling of the simple components by Young diagrams.

(b) There exists an isomorphism between $\mathcal{D}_n(r,q)$ and $\mathcal{D}_n(r,-q^{-1})$ by mapping $T_i(r,q)$ to $T_i(r,-q^{-1})$. This isomorphism maps the simple component $\mathcal{D}_{n,\lambda}(r,q)$ to $\mathcal{D}_{n,\lambda'}(r,-q^{-1})$, where λ' is the Young diagram obtained from the Young diagram λ by interchanging rows with columns. By composing this isomorphism with the isomorphisms under (a), we obtain additional isomorphisms which change the parametrization, e.g. $\mathcal{D}_n(r,q)\cong\mathcal{D}_n(-r^{-1},q)$ is obtained by mapping $T_i(r,q)$ to $-T_i^{-1}(-r^{-1},q)$.

(c) The isomorphisms in (a) and (b) preserve the Markov traces (i.e. the pull-back of the Markov trace under one of these isomorphisms gives the Markov trace of the original algebra).

(d) By uniqueness of the Markov trace, the isomorphisms in (a) and (b) lead to identities for the functions $Q_\lambda(r,q)$ as follows: $Q_\lambda(r,q)=Q_\lambda(-r,-q)=Q_\lambda(r^{-1},q^{-1})=Q_{\lambda'}(r,-q^{-1})=Q_{\lambda'}(-r^{-1},q)$ etc.

The statements above are easily proved (see also e.g [43] Prop. 3.2(c)). It is also immediate that the isomorphisms above are examples of *functorial isomorphisms* which are defined as follows: Let $\bar{\mathcal{D}}_n(r,q)$ and $\bar{\mathcal{D}}_n(r',q')$ be quotients of $\mathcal{D}_n(r,q)$ and $\mathcal{D}_n(r',q')$ respectively. We say that an isomorphism $\Phi:\bar{\mathcal{D}}_n(r,q)\to\bar{\mathcal{D}}_n(r',q')$ is *functorial* if it maps $\langle T_i(r,q)\rangle$ to $\langle T_i'(r',q')\rangle$ for each i with $1\le i<n$; here $\langle a\rangle$ is the subalgebra generated by an element a of an algebra A.

The following Lemma will result in another proof that the representation category of $O(2)$ does not allow any deformations. We will denote by $\bar{\mathcal{D}}_n(r,q)$ the quotient of $\mathcal{D}_n(r,q)$ with respect to the annihilator ideal of $\mathrm{tr}_{\mathcal{D}}$.

Lemma 7.5. *The algebras $\bar{\mathcal{D}}_n(q,q)$ and $\bar{\mathcal{D}}_n(q',q')$ are functorially isomorphic for any $q,q'\in\mathbb{C}$ and any $n\in\mathbb{N}$.*

Proof: One checks easily that $Q_{[n]}(q,q)=2$ for $n>0$, that $Q_{[0]}(q,q)=1=Q_{[1^2]}(q,q)$,and that $Q_\lambda(q,q)=0$ for all other Young diagrams. One deduces from this that $\bar{\mathcal{D}}_n(q,q)\cong\bar{\mathcal{D}}_n(q',q')$ as abstract algebras (see [43], Cor 5.6(b3)). In particular, $\bar{D}_3(q,q)$ is isomorphic to the direct sum of a full 3×3 matrix algebra and a copy of $\mathbb{C}$. Let $p_i^{(\lambda)}$ be the eigenprojection of the element t_i corresponding to the object X_λ, with $\lambda\in\{[0],[1^2],[2]\}$. Using the basis $p_1^{(\lambda)}\circ i_2$ for $\mathrm{Hom}(X,X^{\otimes 3})$, one computes the following matrices

$$(7.9)\quad p_2^{([0])}\ \mapsto\ \frac14\begin{pmatrix}1&1&2\\1&1&2\\1&1&2\end{pmatrix},\ p_2^{([1^2])}\ \mapsto\ \frac14\begin{pmatrix}1&1&-2\\1&1&-2\\-1&-1&2\end{pmatrix},\ \text{and}\ p_2^{([1^2])}\mapsto\ \frac12\begin{pmatrix}1&-1&0\\-1&1&0\\0&0&0\end{pmatrix};$$

this can be done fairly easily by using the dual basis $\left\{\frac{1}{\mathrm{tr}\left(p_1^{(\lambda)}\right)}\,d_2\circ p_1^{(\lambda)}\right\}$ with $\lambda\in\{[0],[1^2],[2]\}$ and the values for $Q_\lambda(q,q)$. The crucial observation now is that these matrices do not depend on q, and hence also the commutation relations between the various $p_1^{(\lambda)}$ and $p_2^{(\lambda)}$, modulo the annihilator ideal of tr. Hence we obtain $\bar{\mathcal{D}}_n(q,q)$ as the quotient of an algebra whose defining relations are independent of q with respect to the annihilator ideal of a trace functional which does not depend on q as well. $\square$

7.9. **Inductive formulas for idempotents.** We will have to study the algebra $\mathcal{D}_n(r,q)$ for values of r and q for which it is not semisimple. This requires more explicit expressions for certain central idempotents. These formulas are special cases for inductive formulas of path idempotents, which have been studied in [32]. However, as we need somewhat more precise information, including the nonsemisimple case, we give a more or less self-contained derivation of the necessary results here.

Let $A\in\mathcal{D}_m$. We shall denote by $A\otimes 1$ (or sometimes just by A, for brevity) the image of A under the usual embedding of $\mathcal{D}_m$ into $\mathcal{D}_{m+1}$ which identifies the generators of $\mathcal{D}_m$ with the first

$m-1$ generators of $\mathcal{D}_{m+1}$. Let $P_{[1^m]}$ denote the central idempotent belonging to $D_{m,[1^m]}$ in the semisimple case. Using the restriction rule (2.1), we can write

$$P_{[1^m]} \otimes 1 = P_{[1^{m+1}]} + P_{[2,1^{m-1}]} + P^{(m+1)}_{[1^{m-1}]}, \tag{7.10}$$

where $P_{[2,1^{m-1}]}$ is an idempotent in $\mathcal{D}_{m+1,[2,1^{m-1}]}$ and $P^{(m+1)}_{[1^{m-1}]}$ is an idempotent in the simple component $D_{m+1,[1^{m-1}]}$. By [32], (2.15), we have

$$P^{(m+1)}_{[1^{m-1}]} = \frac{Q_{[1^{m-1}]}}{Q_{[1^m]}}\ P_{[1^m]}E_m P_{[1^m]}. \tag{7.11}$$

Lemma 7.6. *The idempotents $P_{[1^k]}$ are well-defined if $[m]_q \neq 0$ and $r + q^{1-2m} \neq 0$ for $m = 1, 2,\ \dots\ k$.*

Proof: The claim follows as soon as one has shown the following inductive formula:

$$P_{[1^{m+1}]} = \frac{1}{[m+1]_q}\left(q^m P_{[1^m]} - [m]_q P_{[1^m]}T_m P_{[1^m]} - \frac{[m]_q}{1+rq^{1-2m}} P_{[1^m]}E_m P_{[1^m]}\right). \tag{7.12}$$

Observe that the algebra $\mathcal{D}_{m+1}$ is spanned by elements of the form $A\chi B$, with $A, B \in \mathcal{D}_m$ and $\chi \in \{1, T_m, E_m\}$ (see Theorem 7.3,(d) or, e.g. [43], Prop. 3.2). As $P_{[1^m]}A$ is a scalar multiple of $P_{[1^m]}$ for any $A \in \mathcal{D}_m$, the subalgebra $P_{[1^m]}\mathcal{D}_{m+1}P_{[1^m]}$ is spanned by the three elements $P_{[1^m]}\chi P_{[1^m]}$, with $\chi \in \{1, T_m, E_m\}$. It follows from Lemma 7.2 and from $\mathcal{D}_n/\mathcal{I}_n \cong \mathcal{H}_n$ that we can write

$$P_{[1^{m+1}]} = \frac{1}{[m+1]_q}\left(q^m P_{[1^m]} - [m]_q P_{[1^m]}T_m P_{[1^m]} - \beta P_{[1^m]}E_m P_{[1^m]}\right)$$

for some suitable scalar β. To compute the scalar, we evaluate each side of the equation above under $\mathrm{tr}_{\mathcal{D}}$. Using the Markov property, we obtain

$$\frac{Q_{[1^{m+1}]}}{d(X)^{m+1}} = \frac{1}{[m+1]_q}\left(\frac{q^m Q_{[1^m]}}{d(X)^m} - \frac{[m]_q r Q_{[1^m]}}{d(X)^{m+1}} - \beta\frac{Q_{[1^m]}}{d(X)^{m+1}}\right).$$

Using the explicit formula for $Q_{[1^m]}$ (see Eq 7.6), one can easily solve for β. $\square$

Lemma 7.7. *Assume that $r = q^{m-1}$, with $m > 0$ and that q^2 is a primitive l-th root of unity, $l > m+1$ or $l = \infty$. Then $P_{[1^{m+1}]}$ is well-defined and $P_{[1^{m+1}]} \otimes 1$ is a central minimal idempotent in D_{m+2} modulo the ideal $\mathcal{J}$ generated by $P_{[2,1^{m-1}]}$.*

Proof: It is easy to check that the expressions for $P_{[1^k]}$ in Lemma 7.6 are well-defined for our choice of parameters if $k \leq m+1$; this also implies that $P_{[2,1^{m-1}]}$ is well-defined. As $P_{[2,1^{m-1}]}$ is a linear combination of $P_{[1^m]}\chi P_{[1^m]}$, with $\chi \in \{1, T_m, E_m\}$, it follows from the relations that $E_{m+1}P_{[2,1^{m-1}]}E_{m+1}$ is a scalar multiple of $E_{m+1}P_{[1^m]}$. The scalar can be computed to be equal to $Q_{[2,1^{m-1}]}/Q_{[1^m]}$ by using the Markov property of $\mathrm{tr}_{\mathcal{D}}$. Using this, one easily shows that $P_{[1^{m+1}]}E_{m+1}P_{[1^{m+1}]} \in \mathcal{J}$. As $\mathcal{D}_{m+2}/\mathcal{I}_{m+2} \cong \mathcal{H}_{m+2}$, it follows from Lemma 7.2 and from $\mathcal{D}_n/\mathcal{I}_n \cong \mathcal{H}_n$ that

$$P_{[1^{m+1}]}T_{m+1}P_{[2,1^{m-1}]}T_{m+1}P_{[1^{m+1}]} = \frac{[m]_q}{[m+1]_q}P_{[1^{m+1}]}(T_{m+1}+q^{-1})P_{[1^{m+1}]} + \gamma P_{[1^{m+1}]}E_{m+1}P_{[1^{m+1}]}, \tag{7.13}$$

where γ is some scalar. This implies that also $P_{[1^{m+1}]}(T_{m+1}+q^{-1})P_{[1^{m+1}]}$ is in $\mathcal{J}$. This shows that $P_{[1^{m+1}]} \otimes 1 \equiv P_{[1^{m+2}]} \bmod \mathcal{J}$, if the latter is well-defined.

If q^2 is a primitive $(m+2)$-nd root of unity, we choose as spanning set for the subalgebra $P_{[1^{m+1}]}\mathcal{D}_{m+2}P_{[1^{m+1}]}$ the elements $P_{[1^{m+1}]}$, $P_{[1^{m+1}]}(T_{m+1}+q^{-1})P_{[1^{m+1}]}$ and $P_{[1^{m+1}]}E_{m+1}P_{[1^{m+1}]}$ and show as before that the last two elements are in $\mathcal{J}$. $\square$

Lemma 7.8. *Let q^2 be a primitive l-th root of unity and assume $Q_{[1^k]}(r,q) \neq 0$ for $1 \leq k \leq l$. Then there exists a nilpotent element $N_l \in \mathcal{D}_l(r,q)$ such that $\mathrm{tr}_{\mathcal{D}}(N_l) \neq 0$.*

Proof: We see from Lemma 7.6 that the elements $P_{[1^k]}$ are well-defined for $k < l$, and that also $N_l = [l]_q P_{[1^l]}$ is well-defined. It follows that $N_l^2 = [l]_q N_l = 0$ for our choice of q. Moreover, we have

$$\mathrm{tr}_{\mathcal{D}}(N_l) = [l]_q \frac{Q_{[1^l]}}{d(X)^l}.$$

It is easy to see from Eq 7.6 that $Q_{[1^l]}$ has a pole of order 1 for our choice of q, which cancels with the zero of $[l]_q$ in the formula above. Hence $\mathrm{tr}_{\mathcal{D}}(N_l) \neq 0$ also for q^2 a primitive l-th root of unity. □

Corollary 7.9. *The algebra $\mathcal{D}_l/\mathcal{A}_l$ is not semisimple if q^2 is a primitive l-th root of unity and $Q_{[l]}(r,q) \neq 0$ or $Q_{[1^l]} \neq 0$.*

Proof: If $Q_{[1^l]}(r,q) \neq 0$, we can find an element $N_l \in \mathcal{D}_l(r,q)$ which has nonzero trace (hence also nonzero in the quotient mod $\mathcal{A}_l$) but it is nilpotent. This is not possible in a semisimple algebra. The case with $Q_{[l]}(r,q)$ goes similarly, using one of the isomorphisms in Section 7.8.

The quotient $\mathcal{D}_n/\mathcal{A}_n$ is semisimple for all $n \in \mathbb{N}$ if and only if $Q_{[l]}(r,q) = 0$ and $Q_{[1^l]}(r,q) = 0$; this condition is vacuous for $l = \infty$. The 'only if' part follows from Corollary 7.9. The 'if' part follows from below where we list all the other cases for the parameters r and q. □

8. Identifying End $\left(X^{\otimes n}\right)$

We have seen in the last two sections that there exists a homomorphism Φ from the algebra $\mathcal{D}_n(r,q)$ or $\mathcal{B}D_n(d(X))$ into End $\left(X^{\otimes n}\right)$ given by $T_i \mapsto t_i$ and $E_i \mapsto e_i$. The purpose of this section is to show that this map is surjective.

8.1. **Preliminaries.** We say that two idempotents e and f in an algebra $\mathcal{M}$ are (von Neumann) equivalent, $e \sim f$, if there exist elements u and v in $\mathcal{M}$ such that $e = uv$ and $f = vu$. An idempotent $e \in \mathcal{M}$ is called minimal if there exists for any $a \in \mathcal{M}$ a scalar $\gamma(a)$ such that $eae = \gamma(a)e$. The *multiplicity* $\mathrm{mult}_{\mathcal{M}}(e)$ of an idempotent $e \in \mathcal{M}$ is the maximum number m of idempotents $e_i \in \mathcal{M}$ such that $e_i e_j = 0$ for $i \neq j$ and $e_i \sim e_j$.

Recall that in a semisimple category we can associate to a subobject X of an object Y (i.e. a monomorphism from X into Y) an idempotent p_X in $\mathrm{End}(Y)$ (see e.g. Lemma 3.2). We then define the multiplicity of the subobject X in Y to be equal to the multiplicity of p_X in $\mathrm{End}(Y)$.

Lemma 8.1. *Let $\mathcal{C}$ be a semisimple category, let $Y \in \mathrm{Ob}(\mathcal{C})$ and let $e, f \in \mathrm{End}(Y)$ be two idempotents.*

(a) *The idempotents e and f are equivalent iff $\mathrm{Im}(e)$ and $\mathrm{Im}(f)$ are isomorphic subobjects of Y.*
(b) *Let X be a subobject of $\mathrm{Im}(e)$. Then the multiplicity of X in Y is $\geq \mathrm{mult}_{\mathrm{End}(Y)}(e)$.*

Proof: Follows straightforward from the definitions. □

8.2. Let now $\mathcal{C}$ be a (fusion) category of orthogonal or symplectic type, and let N be the maximum of numbers k for which we have a simple object in $\mathcal{C}$ labeled by a Young diagram of the form $[1^k]$.

Lemma 8.2. *Let $1 \leq m < N$ and assume that $P_{[1^m]}$ and $P_{[1^{m+1}]}$ exist in $\mathcal{D}_{m+1}(r,q)$. Let $p_{[1^k]} = \Phi\left(P_{[1^k]}\right)$ for $k = 1, 2, \ldots, m+1$.*

(a) *If $\mathrm{Im}\left(p_{[1^m]}\right) = X_{[1^m]}$, then $X_{[1^{m+1}]}$ is a subobject of $\mathrm{Im}\left(p_{[1^{m+1}]}\right)$.*
(b) *If moreover $\Phi\left(P_{[2,1^{m-1}]}\right) \neq 0$, then $\mathrm{Im}\left(p_{[1^{m+1}]}\right) = X_{[1^{m+1}]}$.*

Proof: If $m = 1$, the statements are true by definition. Assume now $m > 1$. By induction assumption, $X_{[1^m]}$ is a subobject of $\Phi\left(P_{[1^m]}\right)$; hence $X_{[1^{m+1}]}$ is a subobject of $\Phi\left(P_{[1^m]} \otimes 1\right)$. As $X_{[1^{m+1}]}$ has multiplicity 1 in $X^{\otimes(m+1)}$, the claim in (a) follows from Eq 7.10 and Lemma 8.1.(b).

For part (b), observe that $(p_{[1^m]} \otimes 1)\,\mathrm{End}\,\big(X^{\otimes m+1}\big)(p_{[1^m]} \otimes 1)$ has dimension 3 by Eq 6.1. On the other hand, $\mathrm{tr}_{\mathcal{D}}(P^{(m+1)}_{[1^{m-1}]}) = \dim X_{[1^{m-1}]}/(\dim X)^{m+1} \neq 0$. Using this, our assumption on $\Phi\left(P_{[2,1^{m-1}]}\right)$ and part (a), it follows that the three idempotents on the right hand side of Eq 7.10 have nonzero image under Φ. Hence the claim follows from Eq 6.1. $\square$

8.3. Restrictions for parameters. Let $\mathcal{C}$ and N be as in the previous subsection. Recall that we can choose a fourth root of unity γ such that the eigenvalues of γc are q, $-q^{-1}$ and r^{-1} for suitable values q and r.

Lemma 8.3. *Assume that q^2 is a primitive l-th root of unity, $l \in \mathbb{N} \cup \{\infty\}$ and let $m \in \mathbb{N}$ be such that $Q_{[1^{m+1}]}(r, q) = 0$ and $Q_{[1^k]}(r, q) \neq 0$ for $1 \leq k \leq m$. Then $m < l$ and $m \leq N$.*

Proof: Assume $l \leq m$. By Lemma 7.8, there exists a nilpotent element $N_l \in \mathcal{D}_l$ with $\mathrm{tr}_{\mathcal{D}}(N_l) \neq 0$. Then also $\Phi(N_l)$ is nilpotent and $\mathrm{tr}\left(\Phi(N_l)\right) = \mathrm{tr}_{\mathcal{D}}(N_l) \neq 0$, a contradiction to $\mathrm{End}\left(X^{\otimes l}\right)$ being semisimple.

Now assume that $m > N$. Then it follows from Lemma 7.6 that $P_{[1^{N+1}]}$, $P_{[2,1^{N-1}]}$ and $P^{(N+1)}_{[1^{N-1}]}$ are well-defined. By our assumptions, they also have nonzero trace. From this we could conclude that $\Phi\big((P_{[1^N]} \otimes 1)(\mathcal{D}_{N+1})(P_{[1^N]} \otimes 1)\big)$ would have dimension ≥ 3. This contradicts the fact that $\dim \mathrm{End}\left(X_{[1^N]} \otimes X\right) = 1$ in the orthogonal case and $\dim \mathrm{End}\left(X_{[1^N]} \otimes X\right) = 2$ in the symplectic case (see the remark below Eq 6.1). $\square$

Lemma 8.4.

(a) *If $\mathcal{C}$ has the Grothendieck semiring of an orthogonal group $O(N)$ or of one of its associated fusion categories, then $r = q^{N-1}$ or, if N is odd, $r = -q^{N-1}$.*

(b) *If $\mathcal{C}$ has the Grothendieck semiring of a symplectic group $Sp(N)$ or of one of its associated fusion categories, then $r = -q^{2N+1}$.*

Proof: Let m be as in Lemma 8.3. Assume $m < N$. If $\Phi\big(P_{[2,1^{m-1}]}\big) \neq 0$, then $\mathrm{Im}\,\Phi\big(P_{[1^{m+1}]}\big) = X_{[1^{m+1}]}$ by Lemma 8.2 and $\dim X_{[1^{m+1}]} = Q_{[1^{m+1}]}(r, q) = 0$, which contradicts rigidity, Lemma 5.1.

If $\Phi\big(P_{[2,1^{m-1}]}\big) = 0$, then $X_{[2,1^{m-1}]}$ has to be a subobject of $W = \mathrm{Im}\left(\Phi(P_{[1^{m+1}]})\right)$ by Eq 7.10 ($\mathrm{Im}\left(\Phi(P^{(m+1)}_{[1^{m-1}]})\right)$ is isomorphic to $X_{[1^{m-1}]}$); in particular, $W \cong X_{[1^{m+1}]} \oplus X_{[2,1^{m-1}]}$ is not a simple object. Moreover, $\big(P_{[2,1^{m-1}]} \otimes 1\big) \subset \ker \Phi$ and $\Phi\big(P_{[1^{m+1}]} \otimes 1\big)$ is a central and minimal idempotent in $\Phi(\mathcal{D}_{m+2})$ by Lemma 7.7. By the braiding axioms, we can identify $c_{W,X}$ with an element in $\Phi\big(P_{[1^{m+1}]} \otimes 1\big)\mathcal{D}_{m+2}\big(P_{[1^{m+1}]} \otimes 1\big) \cong \mathbb{C}$. Hence $c_{W,X}$ is a scalar multiple of $1_{W\otimes X}$. As

$$c_{W,X^{\otimes n}} = \big(1_{X^{\otimes n-1}} \otimes c_{W,X}\big) \circ \big(c_{W\otimes X^{\otimes n-1}} \otimes 1_X\big),$$

it follows that $c_{W,X^{\otimes n}}$ is a multiple of the identity for all $n \in \mathbb{N}$. As W is a subobject of $X^{\otimes m+1}$, we also get that $c_{W,W}$ is a multiple of $1_{W\otimes W}$. But then conjugation by $c_{W,W}$ would not permute the factors of $p_{[1^{m+1}]} \otimes p_{[2,1^{m-1}]} \subset \mathrm{End}(W) \otimes \mathrm{End}(W)$, with $p_{[1^{m+1}]}$, $p_{[2,1^{m-1}]}$ the projections onto the submodules of W, contradicting the braiding property. This, together with Lemma 8.3 forces $m = N$.

Using the formulas 7.6, one checks that $m = N$ implies $r = q^{N-1}$, $r = -q^{N-1}$ if N is odd, or $r = -q^{2N+1}$. In case (a) we can rule out $r = -q^{2N+1}$, as in this case also $Q_{[2,1^{N-1}]}\big(-q^{2N+1}, q\big) \neq 0$. In case (b), we can rule out the other cases for r by observing that this would imply $\dim X_{[2,1^{N-1}]} = Q_{[2,1^{N-1}]}\big(-q^{2N+1}, q\big) = 0$, which would contradict rigidity. $\square$

8.4. We can now prove the main result of this section

Theorem 8.5. *Let $\mathcal{C}$ be a tensor category of orthogonal or symplectic type. Then the map Φ : $\mathcal{D}_n(r, q) \to \mathrm{End}\left(X^{\otimes n}\right)$ induced by $T_i \mapsto t_i$ and $E_i \mapsto e_i$ is a well-defined, surjective algebra homomorphism, with the kernel being the annihilator ideal $\mathcal{A}_n$ of the trace $\mathrm{tr}_{\mathcal{D}}$.*

Proof: We have seen in the proof of Lemma 8.4 that a restriction on the number of antisymmetrizations forces r to be equal to $\pm$ a positive power of q. Similarly, it follows from the results in Section 7.8 that a restriction on the number of symmetrizations forces r to be equal to $\pm$ a negative power of q. In particular, if we have restrictions of both the numbers of symmetrizations and antisymmetrizations, the two resulting equalities force q to be a root of unity. It now follows from Section 7.7 that the quotient of $\mathcal{D}_n(r,q)$ modulo the annihilator ideal of the categorical trace coincides with $\mathrm{End}\left(X^{\otimes n}\right)$. $\square$

As an application of this theorem, we can now show that the description of orthogonal and symplectic categories in Section 6.2 was sufficient.

Proposition 8.6. *The Grothendieck semiring of a category of orthogonal or symplectic type is already uniquely determined by the labeling set of its simple objects and the tensor product rules involving the vector representation, see Sections 6.1 and 6.2.*

Proof: Observe that in all our paper we have only used the tensor product rules involving the vector representation to prove the last theorem. By that theorem, any simple object X_λ corresponds to an idempotent p_λ in a quotient $\bar{\mathcal{D}}_n(r,q)$ of $\mathcal{D}_n(r,q)$ for some $n \in \mathbb{N}$. With the simple object X_μ corresponding to an idempotent $p_\mu \in \bar{\mathcal{D}}_m(r,q)$, the multiplicity of X_ν in $X_\lambda \otimes X_\mu$ is now equal to the multiplicity of the idempotent $p_\lambda \otimes p_\mu$ in the simple component of $\bar{\mathcal{D}}_{n+m}(r,q)$ labeled by ν.

It only remains to show that the multiplicity of this idempotent does not depend on the values of the parameters r and q for the various cases (see Sections 6.1, 6.2 and 7.7). A proof probably most suited to our setting goes as follows: A set of minimal idempotents for the algebra $\bar{\mathcal{D}}_n$ was defined in [32], Cor. 2.5 (see also Section 7.9). Strictly speaking, this was only done there for the generic case when $\mathcal{D}_n$ is semisimple, but the proof goes through exactly the same way for $\bar{\mathcal{D}}_n$. More precisely, inductive expressions were given in terms of the generators with coefficients being rational functions in r and q whose singularities are contained in the set of zeros of the dimension functions $Q_\lambda(r,q)$ for our given category. Moreover, explicit matrix representations were determined for the generators of the algebra $\bar{\mathcal{D}}_n(r,q)$ whose matrix entries again are rational functions with singularities as before, see [25], Theorem 6.15.

If $\bar{\mathcal{D}}_n(r,q) \cong \bar{\mathcal{D}}_n(r',q')$ for all n and we are not in the case of a fusion category, we can find a path $(r(t),q(t))$, $0 \le t \le 1$ from (r,q) to (r',q') for which $\bar{\mathcal{D}}_n(r,q) \cong \bar{\mathcal{D}}_n(r(t),q(t))$ for $0 \le t \le 1$, avoiding any possible pole for the matrix representing the idempotent $p_\lambda \otimes p_\mu$. By continuity, the rank of this idempotent hence must be constant in each irreducible representation of $\bar{\mathcal{D}}_n$ if we vary the parameters r and q along our chosen path. For showing the claim in the case of fusion categories, we can find a Galois isomorphism which maps (r,q) to (r',q') (after possibly using some of the reparametrizations mentioned in Section 7.8). This again leaves the rank invariant. $\square$

Remarks: The argument in the last proposition works as well in other cases where the braiding elements of a generating object X of a braided category generate $\mathrm{End}\left(X^{\otimes n}\right)$ for all $n \in \mathbb{N}$. In particular, it can be used for Lie type A and the associated fusion rings (see [23]).

9. Classification of categories of orthogonal or symplectic types

Let $\mathcal{C}$ be a tensor category of orthogonal or symplectic type, and let r and q be the parameters deduced from the eigenvalues of the braiding morphism $c_{X,X}$, see Lemma 6.3. We will show that these parameters will essentially uniquely determine $\mathcal{C}$, up to a few special cases.

9.1. Special Cases. Let us first rule out a few cases for which the following general discussion will not apply: Observe that these include all the possible values of the parameters r and q for which $Q_{[2,1]}(r,q) = 0$ (see Eq 7.7).

(a) It is not possible that q is a root of unity and r is not a root of unity. In this case we would obtain a nilpotent element a in $\mathrm{End}\left(X^{\otimes n}\right)$ for some $n \in \mathbb{N}$ with nonzero categorical trace, which would contradict semisimplicity of $\mathrm{End}\left(X^{\otimes n}\right)$ (see Lemma 7.8 and its corollary).
(b) It is not possible that $r = q^{-1}$ or $r = -q$; this would imply $d(X) = 0$, contradicting rigidity of $\mathcal{C}$.
(c) It is not possible that $r = \pm 1$ and $q \neq \pm 1$. In this case $Q_{[1^2]}(1, q) = 0 = Q_{[2]}(1, q)$, which would contradict rigidity
(d) If $r = q$ or $r = -q^{-1}$ (the $O(2)$-case), we obtain a unique description of $\mathrm{End}\left(X^{\otimes n}\right)$ independent of any parameters r and q (see Lemma 7.5). Hence the diagonal $\mathcal{D}$ in the $O(2)$ case does not depend on q, and there exist exactly two monoidal algebras in this case by Theorem 4.9.
(e) If $r = q^{-3}$ or $r = -q^3$ (the $Sp(1)$-case), $Q_{[1^2]}$ resp $Q_{[2]}$ is equal to 0. Hence in this case we can only obtain a rigid category for which the braiding morphism for the object X has only two distinct eigenvalues. Such categories have been classified in [23] and, for this special case, already before in [14].

9.2. Existence. We have already seen examples of orthogonal or symplectic tensor categories in Section 6.3. The most natural construction in our context uses the tangle category (see [16], [48] [18], [37]). For more details about this construction see [36] and, for the classical case, [10].

An (n, m)-tangle is a collection of $(n + m)/2$ ribbons and an arbitrary number of annuli in $\mathbb{R}^2 \times [0, 1]$; moreover, n ends of the ribbons will be in the intervals $[i - \epsilon, i + \epsilon] \times \{0\} \times \{0\}$, $i = 1, 2, \ldots, n$, and m ends of the ribbons will be in the intervals $[j - \epsilon, j + \epsilon] \times \{0\} \times \{1\}$, $j = 1, 2, \ldots, m$. The concatenation $t_1 \circ t_2$ of an (m, k)-tangle t_1 with an (n, k)-tangle t_2 is given by putting t_1 on top of t_2 and rescaling the z-coordinate.

We want to use tangles to construct monoidal algebras. In order to get finite dimensional morphism spaces, we need some relations between various tangles. These are the Kauffman skein relations (see [19], or also e.g. [43]). To do so consider the following (0,2) and (2,0) tangles

ι π

Figure 1

Here one should think of the ribbon obtained by thickening the lines parallel to the drawing plane. Then $\iota \circ \pi$ is a (2,2) tangle. Further (2,2) tangles are given by 1 (two parallel vertical ribbons) and $\sigma^{\pm 1}$ (two crossing ribbons, where the ± 1 exponent corresponds to the two possible ways of crossing them). We have two possible ways of defining quotients, via the Kauffman skein relations:

$$\sigma - \sigma^{-1} = (q - q^{-1})(1 - \iota \circ \pi) \quad \text{and} \quad \sigma \circ \iota = r^{-1}\iota \tag{9.1}$$

or

$$\sigma + \sigma^{-1} = \sqrt{-1}(q - q^{-1})(1 + \iota \circ \pi) \quad \text{and} \quad \sigma \circ \iota = \sqrt{-1}r^{-1}\iota. \tag{9.2}$$

One can show that the $\mathbb{C}$-span of (0,0) tangles modulo these relations is isomorphic to $\mathbb{C}$. Using this and the morphisms ι and π similar as the morphisms i and d' in Section 2, one defines a trace tr on the set of (n, n)-tangles (see e.g. chapters 2 and 3 in [36]). A $\mathbb{C}$-linear combination a of (n, m)-tangles is called *negligible* if $\mathrm{tr}(ab) = 0$ for any (m, n)-tangle b. Let $\bar{\mathcal{T}}(n, m)_\pm$ be the quotient of the $\mathbb{C}$-span of all (n, m)-tangles modulo the negligible linear combinations of (n, m) tangles with respect to the trace defined by relations 9.1 (for +) or 9.2 (for −). Then it can be checked that, for chosen sign, the collection $(\bar{\mathcal{T}}(n, m)_\pm)_{n,m \in \mathbb{N}}$ is a monoidal algebra of type 2 with $\bar{\mathcal{T}}(n, n)_\pm \cong \mathcal{D}_n(r, q)/\mathcal{A}_n$, whenever the latter is semisimple for all $n \in \mathbb{N}$. From these monoidal algebras, one can construct semisimple categories using the results of Section 3. This has already been shown before in [36], Theorem 8.6. So we have

Proposition 9.1. *There exist categories of orthogonal or symplectic types as quotient categories of the tangle category modulo relations 9.1 or 9.2 for all values r, q for which $\bar{\mathcal{T}}(n,n)_\pm \cong \mathcal{D}_n(r,q)/\mathcal{A}_n$ is semisimple for all $n \in \mathbb{N}$. These cases are all listed in Section 7.7.*

If $q = \pm 1$ (resp $q = \pm i$ in case of relation 9.2), one only obtains interesting categories if also $r = \pm 1$ (resp $r = \pm i$); otherwise $d(X)$ would not be well-defined. Moreover, one needs to add to these relations the additional relation $\pi \circ \iota = d(X)$, with $d(X) \in \mathbb{C}$. In this case, it is often more convenient to consider the resulting structure as a category of graphs (see the work of Brauer [8] and Deligne [10]. Then one obtains monoidal algebras and tensor categories as in the previous proposition (see [10]). Moreover, using the polynomials 7.8, one obtains (see [42], Cor 3.3 and Cor. 3.5)

Proposition 9.2. *There exist orthogonal and symplectic categories obtained as quotient categories of the tangle (or graph) category if $q \in \{\pm 1\}$ or $q \in \{\pm i\}$, with the additional relation $\pi \circ \iota = d(X)$. The resulting category $\mathcal{C}$ has the Grothendieck semiring of* Rep $\big(O(\infty)\big)$ *if $d(X)$ is not an integer. If $d(X)$ is an integer, $\mathcal{C}$ has the Grothendieck semiring of* Rep $\big(O(N)\big)$ *if $d(X) = N$ or, if N is odd, $d(X) = 2 - N$ and it has the Grothendieck semiring of $Sp(N)$ if $d(X) = -2N$.*

9.3. Uniqueness. Let $\mathcal{C}, \tilde{\mathcal{C}}$ be categories of orthogonal or symplectic type with isomorphic Grothendieck semirings, and let r, q resp $\tilde{r}, \tilde{q}$ be the corresponding parameters as determined in Lemma 6.3. We can rule out the special cases considered in Section 9.1; in particular we can assume that $Q_{[2,1]}$ is not zero for these parameters. Let X and $\tilde{X}$ be objects corresponding to the (analogue of the) vector representation in $\mathcal{C}$ and $\tilde{\mathcal{C}}$ respectively. Also, recall that as X generates $\mathcal{C}$, its braiding structure is uniquely determined by $c_{X,X}$.

Theorem 9.3. *Let the notation be as above, and assume that $q \notin \{\pm 1\}$. Then $\tilde{\mathcal{C}}$ is equivalent to $\mathcal{C}$ as monoidal categories if and only if the eigenvalues of $c_{\tilde{X},\tilde{X}}$ can be obtained from the ones of $c_{X,X}$ by changing the braiding and/or the labeling as described in parts (a) and (b) in Section 7.8.*

If $q = \pm 1$, then categories $\mathcal{C}$ and $\tilde{\mathcal{C}}$ constructed as in Prop. 9.2 are equivalent if and only if $d(X) = d(\tilde{X})$ for the additional parameters $d(X)$ and $d(\tilde{X})$, and $c_{X,X}$ and $c_{\tilde{X},\tilde{X}}$ have the same eigenvalues.

Proof: Let $p^{(\lambda)}$ be the eigenprojection of t for X_λ is a subobject of $X^{\otimes 2}$. It is a well-known result for Hecke algebras of type A, that the nonzero eigenvalue of $p_1^{(\lambda)} p_2^{(\lambda)} p_1^{(\lambda)}$ in the summand labeled by $[2,1]$ is equal to $(q+q^{-1})^{-2}$ (see e.g. [41], p. 361). Hence if $\tilde{\mathcal{C}}$ is equivalent to $\mathcal{C}$, we obtain $(q+q^{-1})^{-2} = (\tilde{q}+\tilde{q}^{-1})^{-2}$, which entails $\tilde{q} \in \{\pm q^{\pm 1}\}$. Hence, after changing the braiding structure in $\tilde{\mathcal{C}}$ by replacing $c_{X,X}$ by its negative and/or inverse, if necessary, we can assume $\tilde{q} = q$. It also follows that the quantities $d(X)^2$ and $Q_{[1^2]}$ must be the same for $\mathcal{C}$ and $\tilde{\mathcal{C}}$. Hence we obtain

$$\frac{\tilde{r}-\tilde{r}^{-1}}{q-q^{-1}} + 1 = \pm\left(\frac{r-r^{-1}}{q-q^{-1}} + 1\right). \tag{9.3}$$

If we have a plus sign on the right hand side, it follows that $\tilde{r} \in \{r, -r^{-1}\}$, as claimed. To exclude the minus sign, one uses $Q_{[1^2]}(r,q) = Q_{[1^2]}(\tilde{r},q)$ (see Eq. 7.7) as follows: After substituting the factor $(\tilde{r}-\tilde{r}^{-1})/(q-q^{-1})$, using Eq 9.3, one obtains a second equation in which the only powers of $\tilde{r}$ are $\tilde{r}$ and $\tilde{r}^{-1}$. Solving this linear system in unknowns $\tilde{r}$ and $\tilde{r}^{-1}$, it would follow that $\tilde{r}$ is a rational function of r and q. However, this is not possible for the solution of the quadratic equation 9.3 (in $\tilde{r}$); it is easy to find integer values for r and q for which $\tilde{r}$ is not rational. This finishes the proof of one direction.

On the other hand, assume we have orthogonal or symplectic categories $\mathcal{C}$ and $\tilde{\mathcal{C}}$ with isomorphic Grothendieck semirings, with the parameters (r, q) and $(\tilde{r}, \tilde{q})$ related as in the statement. Hence, after suitable relabeling and change of braiding structure, if necessary, we can assume that the braiding elements $c_{X,X}$ and $c_{\tilde{X},\tilde{X}}$ have the same eigenvalues, for the same components. By Theorem 8.5, this means that both $\mathrm{End}\left(X^{\otimes n}\right)$ and $\mathrm{End}\left(\tilde{X}^{\otimes n}\right)$ are isomorphic to $\bar{\mathcal{D}}_n(r,q) = \mathcal{D}_n(r,q)/\mathcal{A}_n$ for all $n \in \mathbb{N}$. Moreover, under this isomorphism, the tensor operations in $\mathcal{C}$ and $\tilde{\mathcal{C}}$ correspond to the usual embeddings of $\bar{\mathcal{D}}_n(r,q) \otimes \bar{\mathcal{D}}_m(r,q)$ into $\bar{\mathcal{D}}_{n+m}(r,q)$. Hence we obtain an equivalence of the diagonal monoidal algebra generated by X and $\tilde{X}$. By Theorem 4.8 and its corollary, this equivalence extends to the monoidal algebras generated by X and $\tilde{X}$. But then also $\mathcal{C} \cong \tilde{\mathcal{C}}$ by Theorem 3.5. This completes the proof of the theorem if $q \neq \pm 1$.

As the quantity $d(X)$ is independent of the choice of ι and π, equivalent categories of symplectic or orthogonal type must have the same value for $d(X)$. On the other hand, if $q = \pm 1$ for two categories $\mathcal{C}$ and $\tilde{\mathcal{C}}$ of orthogonal or symplectic type for which also $d(X) = d(\tilde{X})$, their diagonal monoidal algebras are given by the Brauer algebras with parameter $d(X) = d(\tilde{X})$, hence are isomorphic. As before, their two possible extensions can be told apart by the eigenvalues of the braiding morphism $c_{X,X}$, by Corollary 4.9.

□

9.4. Main Theorem. Let $\mathcal{C}$ be a tensor category of orthogonal or symplectic type, and let X be the object corresponding to the vector representation.

Theorem 9.4.

(a) *The category $\mathcal{C}$ is completely determined, up to the symmetries mentioned in Theorem 9.3, by the eigenvalues of the braiding morphism $c_{X,X}$, which can be assumed to be of the form $q, -q^{-1}$ and r^{-1} or of the form $iq, -iq^{-1}$ and ir^{-1}, and if $q \in \{\pm 1, \pm i\}$, by the quantity $d(X) = \pi \circ \iota$.*

(b) *The category $\mathcal{C}$ is a fusion category if and only if q is a root of unity and $r = \pm q^n$ for some $n \in \mathbb{Z}$ (see Section 6.2); it is of $O(N)$ or $Sp(N)$ type if and only if $r = \pm q^n$ with n as in Section 6.1 and q not a root of unity or if $q = \pm 1$ and $d(X)$ is an integer, and it is of type $O(\infty)$ if and only if r is not $\pm$ a power of q and q is not a root of unity. Moreover, such categories exist for all possible values of r and q, subject to these conditions, which have not already been excluded in Section 9.1.*

Proof: Part (a) follows from Theorem 9.3. Part (b) now follows from Theorem 8.5 and the results listed in Section 7.7; the existence part follows from Propositions 9.1 and 9.2.

□

Let $\mathcal{E}$ be a fundamental domain for the $\mathbb{Z}/2 \times \mathbb{Z}/2$ action on $\mathbb{C}\backslash\{0\}$ given by $q \to q^{-1}$ and $q \to -q$.

Corollary 9.5. *Braided tensor categories whose Grothendieck semirings are isomorphic to the one of* $\mathrm{Rep}\left(Sp(N)\right)$ *are in 1-1 correspondence with pairs (q, ϵ) where q is a complex number in $\mathcal{E}$ not equal to a root of unity except ± 1 and $\epsilon \in \{\pm 1\}$. The same holds if $Sp(N)$ is replaced by $O(N)$ with N even. For odd N, we have two families of braided tensor categories each of which is labelled by pairs (q, ϵ) as above, which correspond to the cases with $r = q^{N-1}$ and $r = q^{1-N}$.*

Proof: By Theorem 8.5 it suffices to determine all pairs of parameters (r, q) for which $\mathcal{D}_n(r,q)/\mathcal{A}_n \cong \mathrm{End}\left(X^{\otimes n}\right)$ for all $n \in \mathbb{N}$. Using the symmetries in Section 7.8 and the results in [43], Theorem 6.4 (see Section 7.7), one shows first that we can assume the parameters (r, q) to be of the form (q^m, q), with $q \in \mathcal{E}$. Again using [43], Theorem 6.4 (and [42], Cor. 3.5 for $q = 1$), one can read off which exponent belongs to which group. □

Remark : The categories whose Grothendieck semirings are isomorphic to the ones of a symplectic or an even-dimensional orthogonal group as well as one of the two families in the odd-dimensional orthogonal case are closely related to the corresponding Drinfeld-Jimbo quantum groups. The second family of categories in the odd-dimensional case seems to be different. For instance, it is not possible to obtain positive dimensions for all objects, for any choice of parameters, even after changing the quantity α (see Lemma 5.4) for the dimension function.

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