REPRESENTATIONS OF THE BRAID GROUP B_3 AND OF SL(2,**Z**)

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We give a complete classification of all simple representations of B_3 for dimension $d \leq 5$ over an algebraically closed field K of any characteristic. To describe our result in detail, recall that B_3 is given by generators σ_1 and σ_2 which satisfy the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. Moreover, it is well-known that the center of B_3 is generated by $\zeta = (\sigma_1\sigma_2)^3$. It is easy to see that ζ acts on a simple d-dimensional B_3- module via the scalar δ which satisfies the equation $\delta^d = \det(A)^6$, where A is the linear endomorphism via which σ_1 acts on V. Our main result states that the eigenvalues of A and the scalar δ completely determine a simple representation of B_3 for dimension ≤ 5, up to equivalence; for $d \leq 3$ it is uniquely determined by the eigenvalues of A. Moreover, such simple representations exist if and only if the eigenvalues do not belong to the zero set of certain polynomials in the eigenvalues and δ which we list explicitly (see Proposition 2.8, Section 2.10 and 2.11, Remark 4).

One of the motivations for this paper was studying braided tensor categories by analysing braid representations. The categories under consideration have a Grothendieck semiring isomorphic to the one of a semisimple Lie group. It turns out that in this context it suffices to classify representations of B_3 up to dimension 5. Indeed, as an application, we obtain uniform formulas for the categorical dimensions of objects in the second tensor power of the adjoint representation in braided tensor categories closely related to the conjectured series of exceptional Lie algebras, as proposed by Deligne and Vogel.

Our approach in this paper is quite elementary: we first show that assuming a certain triangular form of the matrices A and B of the generators of B_3, the braid relation reduces to checking the values of certain coefficients of the matrix BA. We then show that for dimension $d \leq 5$ one can always assume such a triangular form, and that the matrix coefficients are determined by the eigenvalues and δ, up to certain renormalizations. These explicit representations are then used to compute polynomials which determine whether the representation is simple or not. Moreover, we show in the last section how these polynomials can be used to compute categorical dimensions of objects in braided categories, and we give explicit formulas for them.

After having completed this research we learned of other approaches towards classifying braid representations, notably via local systems by N. Katz, and via quivers. These approaches as well as the case with $d \geq 6$ are discussed in the end of chapter 2.

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1.1. Let B_3 be Artin's braid group, given by generators σ_1 and σ_2 and the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. It is well-known that B_3 maps surjectively onto $SL(2,\mathbf{Z})$, e.g. via the map

$$\sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

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It is easy to check that this homomorphism maps

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \ \mapsto\ S \ = \ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_1\sigma_2 \ \mapsto\ U \ = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Moreover, the center of $SL(2, \mathbf{Z})$ is equal to ± 1, and the corresponding elements $\bar{S}$ and $\bar{U}$ in the quotient $PSL(2, \mathbf{Z})$ have orders 2 and 3 respectively. It is known that $PSL(2, \mathbf{Z})$ is isomorphic to the free product $\mathbf{Z}_2 * \mathbf{Z}_3$ of a cyclic group of order 2 with a cyclic group of order 3, and that the isomorphism can be chosen such that $\bar{S}$ and $\bar{U}$ are the generating elements of these cyclic groups. Finally, the center Z of B_3 is generated by $(\sigma_1\sigma_2)^3$, and $B_3/Z \cong PSL(2, \mathbf{Z})$.

1.2. In the following we assume A and B to be invertible $d \times d$ matrices over the algebraically closed field K satisfying $ABA = BAB$. Hence $\sigma_1 \mapsto A$, $\sigma_2 \mapsto B$ defines a representation of B_3. Let $\lambda_1, \ ... \ \lambda_d$ be the eigenvalues of A (with λ_i not necessarily distinct from λ_j for $i \neq j$) and define $\delta = det(A)^{6/d} = (\prod_{i=1}^d \lambda_i)^{6/d}$.

Lemma (a) Conjugation via ABA maps A to B and B to A.

(b) $ABA(AB)^{-1} = B$ and $BAB(BA)^{-1} + A$

(c) $(ABA)^2 = \delta 1$, and $(ABA)^{-1} = \delta^{-1}ABA$ if A and B generate the full $d \times d$ matrix ring.

(d) For any choice $\delta^{1/6}$ of a sixth root of δ the map $\sigma_1 \mapsto \delta^{-1/6}A$, $\sigma_2 \mapsto \delta^{-1/6}B$ defines a representation of B_3 and of $PSL(2, \mathbf{Z}) \cong B_3/Z$.

(e) If $\{e_1, e_2, \ ... \ e_d\}$ is a basis of eigenvectors of A, and $b_i = ABAe_i$, then $\{b_1, b_2, \ ... \ b_d\}$ is a basis of eigenvectors of B.

Proof It follows from the braid relation that $BAB^{-1} = A^{-1}BA$, from which one easily deduces claims (a) and (b). For (c), observe that $(\sigma_1\sigma_2\sigma_1)^2 = (\sigma_1\sigma_2)^3$ is in the center of B_3. As we assume an irreducible representation of B_3, the center has to act via a scalar matrix, say $\lambda 1$. Comparing determinants (using $det(A) = det(B)$, we get $\lambda^d = \det(A)^6$. Statement (e) follows from (a); (d) is clear.

1.3. We say that the $d \times d$ matrices A and B are in *ordered triangular* form if A is an upper triangular matrix with eigenvalue λ_i as i-th diagonal entry, and B is a lower triangular matrix with eigenvalue λ_{d+1-i} as i-th diagonal entry.

Lemma Assume A and B are in ordered triangular form, satisfying $ABA = BAB$. Then

(a) Any $d \times d$ matrix $F = (f_{ij})$ such that $FBF^{-1} = A$ has matrix entries $f_{ij} = 0$ for $i + j > d + 1$. This holds, in particular, for $F = BA$.

(b) Any matrix $d \times d$ $E = (e_{ij})$ such that $EAE^{-1} = B$ has matrix entries $e_{ij} = 0$ for $i + j < d + 1$. This holds, in particular, for $E = AB$.

Proof. Let $\{w_1, w_2, \ ... \ w_d\}$ be the standard basis for K^d. By assumption, the eigenspace of A corresponding to eigenvalues $\lambda_1, \ ... \ \lambda_r$ is equal to span$\{w_1, w_2, \ ... \ w_r\}$, for $r = 1, 2, \ ... \ d$, while the corresponding eigenspace for B is equal to span $\{w_{\bar{1}}, w_{\bar{2}}, \ ... \ w_{\bar{r}}\}$, where $\bar{i} = d + 1 - i$. Hence F maps span$\{w_1, w_2, \ ... \ w_r\}$ onto span$\{w_{\bar{1}}, w_{\bar{2}} \ ... \ w_{\bar{r}}\}$, from which we deduce the triangular shape of F. The second statement in (a) follows from Lemma 1.2(b). Statement (b) is proved similarly.

Proposition Assume A and B are in ordered triangular form, satisfying $ABA = BAB$. Let $\bar{i} = d+1-i$. After rescaling our basis vectors, if necessary, we can assume

(a) ABA is a skew diagonal matrix, with $(ABA)_{i,\bar{i}} = (-1)^{i+1}\delta^{1/2}$,

(b) $b_{ij} = (-1)^{i+j}a_{\bar{i},\bar{j}}$,

(c) $\sum_{k=1}^{i}(-1)^{k+i}a_{\bar{i},\bar{k}}a_{k,\bar{i}} = (BA)_{i,\bar{i}} = (-1)^{i+1}\delta^{1/2}/\lambda_i$,

(d) If $i+j > d+1$, $\sum_{k=1}^{min(i,j)}(-1)^{k+i}a_{\bar{i},\bar{k}}a_{k,j} = (BA)_{ij} = 0$

(e) Statements (a)-(d) also hold for the coefficients of the matrices $A' = DAD^{-1}$ and $B' = DBD^{-1}$ if D is a diagonal matrix with $d_{ii} = d_{\bar{i}\bar{i}} \neq 0$. In particular, if the entries a_{id} for $1 < i \leq [(d+1)/2]$ are nonzero, we can choose arbitrary nonzero numbers for them.

Proof. Statements (a) and (b) follow from the previous Lemma and Lemma 1.2(a) and (c). Statements (c) and (d) follow from explicit matrix computations, exploiting the triangular shapes of A, B and BA, and using statement (a). For (e), observe that conjugation by D does not change ABA, and also leaves A' and B' in ordered triangular form. One can now prove statements (a)-(d) for A' and B' as we did before for A and B.

Corollary (a) $a_{1d} = \delta^{1/2}/\lambda_1\lambda_d$,,

(b) $a_{2,d-1} = (\lambda_2 a_{d-1,d}a_{1,d-1} - \delta^{1/2})/\lambda_2\lambda_{d-1}$,

Proof. By Proposition, (c), we have $\delta^{1/2}/\lambda_1 = (BA)_{1d}$, with the latter being equal to $a_{1d}\lambda_d$. Statement (b) follows similarly, using Proposition (c).

1.4. Proposition Let A be an upper triangular matrix with eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_d$ down the diagonal, and let S be a skew-diagonal matrix with $S^2 = c1$ for some constant c such that $B = SAS^{-1}$. Moreover, assume

(a) $(BA)_{ij} = 0$ for $i+j > d+1$,

(b) $\lambda_i(BA)_{i\bar{i}} = cs_{i,\bar{i}}$

Then A and B satisfy the braid relations.

Proof. Observe that S^{-1} is a scalar multiple of S. Hence $B = S^{-1}AS = SAS^{-1}$. We get from this $AB = S(BA)S^{-1}$, hence $(AB)_{ij} = 0$ for $i+j < d+1$. Exploiting the triangular shapes of the matrices, we get that $A(BA)$ is upper skew-triangular and $(AB)A$ is lower skew-triangular. Hence ABA is of skew-diagonal shape, with $(ABA)_{i,\bar{i}} = \lambda_i(BA)_{i,\bar{i}} = cs_{i\bar{i}}$, and ABA commutes with S. Hence

$$BAB = S(ABA)S^{-1} = ABA.$$

1.5. An example Let V be a $(d+1)$-dimensional vector space, with a basis labeled by $0, 1, \ldots d$, and let $\lambda_0, \lambda_1, \ldots \lambda_d$ be parameters satisfying $\lambda_i\lambda_{d-i} = c$ for a fixed constant c. Only for this subsection we define $\bar{i} = d-i$. Then we get a $[\frac{d+1}{2}]$ parameter family of representations of B_3 via the matrices

$$A = \left(\binom{\bar{i}}{\bar{j}} \lambda_j \right)_{ij}, \quad B = \left((-1)^{i+j} \binom{i}{j} \lambda_{\bar{i}} \right)_{ij}.$$

To prove this, it satisfies to check the conditions of Proposition 5, with S being the skew-diagonal matrix defined by $s_{ij} = (\delta_{i,\bar{j}}(-1)^i \lambda_{\bar{i}})$. This can be fairly easily checked, using the identity

$$\sum_{k=0}^{d} (-1)^{i+k} \binom{i}{k} \binom{\bar{k}}{\bar{j}} = (-1)^i \binom{d-i}{d-j} = (-1)^i \binom{\bar{i}}{\bar{j}}.$$

This identity is well-known, and can be easily proved by induction on i.

2. Representations of B_3 of dimension $d \leq 5$.

2.1 In the following let V be a B_3-module over the field K. Let $\{e_1, e_2, \ ... \ e_d\}$ be a basis of generalized eigenvectors of A, and let $\{b_1, b_2, \ ... \ b_d\}$ be a basis of generalized eigenvectors of B with $b_i = ABAe_i$ for $i = 1, 2, \ ... \ d$. We shall always assume that if A and B have Jordan blocks, the labeling is chosen such that they are upper triangular. As a consequence of our assumptions and Lemma 1.2, there exist scalars c_{ij}, $1 \leq i, j \leq d$ such that

$$b_i = \sum_{j=1}^{d} c_{ji} e_j \quad \text{and} \quad e_i = \delta^{-1} \sum_{j=1}^{d} c_{ji} b_j.$$

As $(ABA)^2 = \delta 1$, the matrix ABA is diagonalizable. We will usually refer to an eigenvalue of ABA just as $\delta^{1/2}$, not specifying which root we choose. We shall also consider the subspaces

$$W = \text{span}\{e_1, e_2, \ ... \ e_{d-1}\} \ \cap \ \text{span}\{b_1, b_2, \ ... \ b_{d-1}\}$$

$$W' = \bigcap_{i \in \mathbf{N}} (AB)^i (\text{span}\{e_1, e_2, \ ... \ e_{d-1}\}).$$

Observe that $W' \subset W$, as $ABe_i = ABAA^{-1}e_i = \lambda_i^{-1} b_i$. Moreover, W is invariant under ABA and W' is invariant under AB. Also observe that W has codimension ≤ 2 and W' has codimension ≤ 3, being the intersection of 2 resp. 3 subspaces of V with codimension 1. We shall need the following simple observations:

(i) Each of the sets $\{\sigma_1, \sigma_2\}$, $\{\sigma_1, \sigma_1\sigma_2\}$ or $\{\sigma_1\sigma_2\sigma_1, \sigma_1\sigma_2\}$ generates B_3.

(ii) Let I be a subset of $\{1, 2, \ ..., \ d\}$. If $V' = \text{span}\{e_i, \ i \in I\} = \text{span}\{b_i, \ i \in I\}$, then V' is a B_3-submodule.

(iii) If W has codimension 1, it would coincide with $\text{span}\{e_1, e_2, \ ... \ e_{d-1}\}$ and with $\text{span}\{b_1, b_2, \ ... \ b_{d-1}\}$, and therefore would be a B_3-submodule, by (ii).

(iv) If both W and W' have codimension 2, they would coincide; this space would be invariant under both ABA and AB, from which one easily deduces that it is a B_3-submodule, by (i).

2.2. Proposition Let V be a simple B_3-module of dimension $d \leq 5$. Then an eigenvector of A, say e_i can not be contained in a proper subspace of V which is invariant under B and contains $b_i = ABAe_i$.

Proof. We choose the labeling of (generalized) eigenvectors so that e_1 is contained in a B-invariant subspace spanned by generalized eigenvectors b_1, ..., b_{d-1}. Let W and W' be as in 2.1. By 2.1(iii) and (iv), the claim follows immediately unless W has codimension 2 and W' has codimension 3. As W has at least dimension 1, containing e_1, we get the claim for $d = 2$ immediately.

$d = 3$: Here W has to be 1-dimensional, containing both e_1 and b_1, i.e. it is an eigenspace of both A and B.

$d = 4$: By assumption, W contains both e_1 and b_1 and has dimension 2. If e_1 and b_1 are linearly dependent, the claim follows from (2.1)(ii). Hence we can assume them to be a basis for W. As dim $W' = 1$, there exists an eigenvector $v = \alpha e_1 + \beta b_1$ of AB in W'. We can assume $\beta \neq 0$, as otherwise v would be an eigenvector of both A and AB, from which the claim would follow by 2.1(i). But then

$$\mu(\alpha e_1 + \beta b_1) = AB(\alpha e_1 + \beta b_1) = \lambda_1^{-1} b_1 + \beta\lambda_1 Ab_1,$$

where μ is the eigenvalue of v. We deduce from this $Ab_1 \in W$, and also $\delta Be_1 = BABAb_1 \in BAB(W) = W$. Hence W is invariant under both A and B.

$d = 5$: We can assume W to have a basis $\{e_1, b_1, x\}$, with x an eigenvector of ABA with eigenvalue $\delta^{1/2}$ (see 2.1). As AB is diagonalizable, we can choose a basis of eigenvectors $v_i = \alpha^{(i)} e_1 + \beta^{(i)} b_1 + \gamma^{(i)} x$, $i = 1, 2$ for W'. We have $ABv_i = w_i + \alpha^{(i)}\lambda_1^{-1} b_1$, with $w_i = \lambda_1\beta^{(i)} Ab_1 + \gamma^{(i)} ABx$, for $i = 1, 2$. As ABv_i and b_1 are in W, so is w_i, for $i = 1, 2$.

Case 1 : If w_1 and w_2 are linearly dependent, then there exist scalars ν_1 and ν_2 such that $\nu_1 v_1 + \nu_2 v_2 \neq 0$ and $\nu_1 w_1 + \nu_2 w_2 = 0$. But then $AB(\nu_1 v_1 + \nu_2 v_2) \in W'$ is a nonzero multiple of b_1; in particular, $b_1 \in W'$. But then also

$$\delta\lambda_1 e_1 = \delta Ae_1 = A(BAB)b_1 = (AB)^2 b_1 \in W'.$$

Hence W' is spanned by e_1 and b_1, and therefore is also invariant under ABA.

Case 2 : If w_1 and w_2 are linearly independent, then also ABx and Ab_1 are in W. We conclude $Be_1 \in W$ as in the case $d = 4$. Moreover, also $\delta^{1/2} B^{-1} x = B^{-1} BABx = ABx \in W$, i.e. $B^{-1}x$ is a linear combination of e_1, b_1 and x. If the coefficient of x is not equal to 0 in this linear combination, we can multiply it by B and solve for Bx, which shows that it is in W. In this case, W is invariant under both B and BAB. If $B^{-1}x$ is a linear combination of only e_1 and b_1, we can multiply this linear combination by AB to obtain for Ax a linear combination in b_1 and $Ab_1 \in W$. Hence W is invariant under A and ABA. The claim follows, using 2.1(i).

Corollary Let V be a simple B_3-module with dimension $d \leq 5$. Then the minimal polynomial of A coincides with its characteristic polynomial.

Proof. Let $b \in V$, let $\tilde{S} = \mathrm{span}\{A^i b,\ i = 0, 1,\ \dots\}$, and let λ be an eigenvalue of A. Then it is well-known that the intersection of the eigenspace of the eigenvalue λ (for A)

with $\tilde{S}$ has at most dimension 1. To see this directly, let p_λ be the projection onto the generalized eigenspace of A for the eigenvalue λ, with the kernel being the direct sum of the generalized eigenspaces for the other eigenvalues. Then we get

$$p_\lambda \tilde{S} = \text{span}\{A^i p_\lambda b, i = 0, 1, \ ... \} = \text{span}\{(A-\lambda)^i p_\lambda b, i = 0, 1, \ ... \}.$$

The claim now follows from the fact that the nonzero elements in $\{(A-\lambda)^i p_\lambda b, i = 0, 1, \ ... \}$ are linearly independent, and A acts as a full Jordan block on their span.

If the minimal polynomial of A does not coincide with the characteristic polynomial, there exists an eigenvalue, say λ_1, whose eigenvectors span a subspace E of dimension at least 2. If there is an eigenvalue distinct from λ_1, say λ_j, pick an eigenvector b_j of B belonging to λ_j. Then $\text{span}\{A^s b_j, s = 0, 1, \ ... \ d-2\}$ forms an A-invariant subspace S' such that $\dim S' \cap E \leq 1$, as proved in the last paragraph. Let S be the subspace generated by S' and the eigenvectors of A with eigenvalue λ_j. Then also $\dim S \cap E \leq 1$, i.e. S is a proper A-invariant subspace of V which contains both b_j and BAb_j. Hence V could not be a simple B_3-module.

If A only has one eigenvalue, $\dim E \geq 2$ implies that we have at least 2 different Jordan blocks. Let b_j be an eigenvector of B belonging to a block of maximum length. Let $S' = \text{span}\{A^s b_j, s = 0, 1, \ ... \ d-2\}$, which is a proper subspace of V by assumption. If S' did not contain $e_j = BAb_j$, it would not contain the whole Jordan block of e_j either. Hence the space S spanned by S' and $\{e_j\}$ is a proper A-invariant subspace containing both e_j and b_j, and hence is a B_3-submodule. If S' does contain e_j, it would be a proper B_3-submodule by the same argument.

2.3 Lemma Let e_1 and e_2 be either 2 eigenvectors of A with eigenvalues λ_1 and λ_2, or let them be generalized eigenvectors with $Ae_1 = \lambda e_1$ and $Ae_2 = \lambda e_2 + e_1$.

(a) Let $V' = \text{span}\{e_1, e_2, b_1, b_2\}$. If $\dim V' \leq 3$, then there exists a nonzero B_3-invariant subspace of V of dimension ≤ 3.

(b) Let $d = 5$ and let $V'' = \text{span}\{e_1, e_2, e_3, b_1, b_2\}$, with e_3 a generalized eigenvector belonging to e_1 or e_2, or an eigenvector. If $\dim V'' \leq 4$, it contains a nonzero B_3-invariant subspace.

Proof. The claim in (a) follows from 2.1(ii) if dim $V' = 2$. Hence we can assume dim $V' = 3$. Let $W = \text{span}\{e_1, e_2\} \cap \text{span}\{b_1, b_2\}$. Then dim $W = 1$, as otherwise $\{e_1, e_2, b_1, b_2\}$ would be linearly independent. In particular, W is spanned by an eigenvector $w = \alpha_1 e_1 + \alpha_2 e_2$ of ABA. Let us assume first that both e_1 and e_2 are eigenvectors. If $\lambda_1 = \lambda_2$, w would be a common eigenvector of both A and ABA, and we are done. Hence we can assume $\lambda_1 \neq \lambda_2$. We can also assume $\alpha_2 \neq 0$ in the expression for w, as otherwise w would be an eigenvector of both A and ABA. Moreover, we also have $\delta^{1/2} w = ABAw = \alpha_1 b_1 + \alpha_2 b_2$. We now compute

$$ABw = B^{-1}BAB(\alpha_1 e_1 + \alpha_2 e_2) = \alpha_1 \delta^{1/2} w + \alpha_2(\lambda_2^{-1} - \lambda_1^{-1}) b_2,$$

$$(AB)^2 w = \delta^{1/2} A(\alpha_1 e_1 + \alpha_2 e_2) = \delta^{1/2}(\lambda_1 w + \alpha_2(\lambda_2 - \lambda_1) e_2).$$

The coefficients of e_2 and b_2 in the expressions above are nonzero. Hence the vectors $\{e_2, b_2, w\}$ and $\{w, \ ABw, \ (AB)^2 w\}$ are bases for the same subspace. This proves that it is both invariant under AB (second basis, as $(AB)^3w = \delta w$) and under ABA (first basis). This proves statement (a) if both e_1 and e_2 are eigenvectors. If e_2 is a generalized eigenvector, we compute

$$ABw = \delta^{1/2}\lambda^{-1}w - \delta^{1/2}\alpha_2\lambda^{-2}b_1, \quad (AB)^2w = \delta^{1/2}\lambda w + \delta^{1/2}\alpha_2 e_1.$$

One shows as before that $\mathrm{span}\{w,\ ABw,\ (AB)^2w\} = \mathrm{span}\{e_1, b_1, w\}$, from which one deduces the claim.

Observe that $V' \subset V''$. Statement (b) follows immediately from (a) if $\dim V' < 4$. Hence we can assume $\dim V' = 4$ and $V' = V''$, which therefore is invariant under ABA. Moreover, the space $U = \mathrm{span}\{e_1, e_2, e_3\} \cap \mathrm{span}\{b_1, b_2, b_3\}$ has at least dimension 2, being the intersection of 2 subspaces of V' with codimension 1; we can assume $\dim U = 2$ as otherwise U would be an invariant subspace. Hence it contains 2 linearly independent eigenvectors w_1 and w_2 of ABA. Moreover, if $w_1 = \sum \alpha_j e_j$, we can assume $\alpha_3 \neq 0$; otherwise $w_1 \in \mathrm{span}\{e_1, e_2\} \cap \mathrm{span}\{b_1, b_2\}$ which is 0 (the vectors e_1, e_2, b_1, b_2 are a basis for V'). Now the equality $ABAw_1 = \delta^{1/2}w_1$ implies

$$BAw = B(\sum_{j=1}^{3}\alpha_j\lambda_j e_j) = \sum_{j=1}^{3}\delta^{1/2}\alpha_j\lambda_j^{-1}e_j = \delta^{1/2}A^{-1}w. \qquad (*)$$

We want to show that we can assume $e_3 \notin U$. If e_3 is a generalized eigenvector belonging to e_1 with $e_3 \in U$, we can replace it by $e_3 + \tau e_1$ for any scalar $\tau \in K$. This new vector would only be in U for any choice of τ if $e_1 \in U$. But then U would be spanned by e_1 and e_3, and would be a B_3-submodule, being invariant under A and ABA. If e_3 is an eigenvector in $U \subset \mathrm{span}\{b_1, b_2, b_3\}$, the claim follows from Prop. 2.2.

So we can assume that $X = \mathrm{span}\{e_3, w_1, w_2\}$ coincides with $\mathrm{span}\{e_1, e_2, e_3\}$. If e_4 and e_5 were eigenvectors with same eigenvalue λ, then either $ABAe_3 \in X$ (which would make X invariant under both A and ABA), or $\mathrm{span}(\{ABAe_3\} \cup X)$ would be equal to $\mathrm{span}(\{\tilde{e}_4\} \cup X)$, where $\tilde{e}_4$ is a linear combination of e_4 and e_5, and, in particular, it is an eigenvector of A. Using these 2 different spanning sets, we see that we would obtain a 4-dimensional subspace invariant under both A and ABA.

Hence we can assume that V/X has at most 2 A-invariant 1-dimensional subspaces, and, similarly $V/\mathrm{span}\{b_1, b_2, b_3\}$ has at most 2 B-invariant 1-dimensional subspaces. Expanding the vectors in the equation $(*)$ as a linear combination of $\{e_3, w_1, w_2\}$, and observing that $w_1, w_2 \in \mathrm{span}\{b_1, b_2, b_3\}$, we obtain that Be_3 is congruent to a multiple of e_3 modulo $\mathrm{span}\{b_1, b_2, b_3\}$. But this would imply that e_3 together with b_1, b_2, b_3 spans a B-invariant subspace of dimension ≤ 4. Statement (b) now follows from Proposition 2.2, if e_3 is an eigenvector. If e_3 is a generalized eigenvector belonging to, say e_1, we could show as before that also $e_3' = e_3 + \tau e_1$ is in a 4-dimensional B-invariant subspace containing b_1, b_2 and b_3. As this is true for any $\tau \in K$, and there are only at most 2 such subspaces, we obtain that also e_1 itself is in a B-invariant subspace containing b_1. The claim follows from Proposition 2.2.

2.4. Proposition Let V be a simple B_3-module with dimension ≤ 5. Then there exists a basis of V with respect to which A and B act in ordered triangular form (see 1.3).

Moreover, this is possible for any labeling of the generalized eigenvectors as long as A appears in upper triangular Jordan form.

Proof. To construct such a basis, we can always assume at least one eigenvector e_1 for A and b_1 for B, and at least one more (generalized) eigenvector e_2 as in Lemma 2.3. We define $w_1 = e_1$ and $w_d = b_1$. For $d \geq 3$, observe that $c_{d1} \neq 0$ by Proposition 2.2, with c_{ij} as in 1.3, for $1 \leq i, j \leq d$. Hence we can define $w_2 = e_2 - (c_{d2}/c_{d1})e_1$, which is both in $\mathrm{span}\{e_1, e_2\}$ and in $\mathrm{span}\{b_1, \ ..., \ b_{d-1}\}$. For $d \geq 4$, define $w_{d-1} = ABAw_2$, which is in $\mathrm{span}\{b_1, b_2\}$ and in $\mathrm{span}\{e_1, \ ..., \ e_{d-1}\}$. Finally, if $d = 5$, we can express b_3 as a linear combination of $\{e_1, e_2, e_3, b_1, b_2\}$, by Lemma 2.3. Hence there exist scalars α_1 and α_2 such that $w_3 = b_3 - \alpha_1 b_1 - \alpha_2 b_2$ is in $\mathrm{span}\{e_1, e_2, e_3\} \cap \mathrm{span}\{b_1, b_2, b_3\}$. By construction, it follows that

$$w_i \in \mathrm{span}\{e_1, e_2, \ ... \ e_i\} \cap \mathrm{span}\{b_1, b_2, \ ... \ b_{d+1-i}\} \quad \text{for } 1 \leq i \leq 3 \qquad (*)$$

It follows from $(*)$ that $\mathrm{span}\{w_1, w_2, \ ... \ w_i\} \subset \mathrm{span}\{e_1, e_2, \ ... \ e_i\}$ and that $\mathrm{span}\{w_{d+1-i}, w_{d+2-i}, \ ... \ w_d\} \subset \mathrm{span}\{b_1, b_2, \ ... \ b_i\}$. Let us check equality for $d = 5$. This follows by construction for $i \leq 3$. But then $\mathrm{span}\{w_1, w_2, w_3\} = \mathrm{span}\{e_1, e_2, e_3\}$ and $\mathrm{span}\{w_4, w_5\} = \mathrm{span}\{b_1, b_2\}$. The linear independence of the w_i's now follows from Lemma 2.3(b). Hence the inclusions below $(*)$ actually are equalities, from which easily deduces the triangular forms of A and B by induction on i. The cases $d < 5$ are similar and easier to check.

2.5. Proposition Let V be a simple B_3-module of dimension d with $d = 2, 3$. Then there exists a basis for V for which A and B acts via the matrices

$$A = \begin{pmatrix} \lambda_1 & \lambda_1 \\ 0 & \lambda_2 \end{pmatrix}, \qquad B = \begin{pmatrix} \lambda_2 & 0 \\ -\lambda_2 & \lambda_1 \end{pmatrix} \quad \text{for } d = 2$$

$$A = \begin{pmatrix} \lambda_1 & \lambda_1\lambda_3\lambda_2^{-1} + \lambda_2 & \lambda_2 \\ 0 & \lambda_2 & \lambda_2 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad B = \begin{pmatrix} \lambda_3 & 0 & 0 \\ -\lambda_2 & \lambda_2 & 0 \\ \lambda_2 & -\lambda_1\lambda_3\lambda_2^{-1} - \lambda_2 & \lambda_1 \end{pmatrix} \quad \text{for } d = 3.$$

Proof. For $d = 2$: By Proposition 2.4, we can assume A and B in ordered triangular form, with only the nonzero off-diagonal entries to be computed. Rescaling one of the basis vectors, we can assume $a_{12} = \lambda_1$. We obtain $b_{21} = -\lambda_2$ from the braid relation $ABA = BAB$.

For $d = 3$: Again, by Proposition 2.4, we only need to compute the nonzero off-diagonal entries of A and B, which are in ordered triangular form. By Corollary 1.3, we can assume $a_{13} = \lambda_2 = b_{31}$. If $a_{23} = 0$, then also b_{21}, by Proposition 1.3(b). But then $\mathrm{span}\{w_1, w_3\}$ would be a subspace invariant under both A and B, contradicting V being simple. Hence we can assume $a_{23} = \lambda_2 = -b_{21}$, by Prop. 1.3(e). Finally, we compute from $(BA)_{33} = 0$ that $b_{32} = -\lambda_1\lambda_3\lambda_2^{-1} - \lambda_2$.

2.6. Proposition Let V be a 4-dimensional B_3-module, and let $D = \sqrt{\lambda_2\lambda_3/\lambda_1\lambda_4}$. Then we can find a basis for V with respect to which we get the matrices

$$A = \begin{pmatrix} \lambda_1 & (1+D^{-1}+D^{-2})\lambda_2 & (1+D^{-1}+D^{-2})\lambda_3 & \lambda_4 \\ 0 & \lambda_2 & (1+D^{-1})\lambda_3 & \lambda_4 \\ 0 & 0 & \lambda_3 & \lambda_4 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$$

$$B = \begin{pmatrix} \lambda_4 & 0 & 0 & 0 \\ -\lambda_3 & \lambda_3 & 0 & 0 \\ D\lambda_2 & -(D+1)\lambda_2 & \lambda_2 & 0 \\ -D^3\lambda_1 & (D^3+D^2+D)\lambda_1 & -(D^2+D+1)\lambda_1 & \lambda_1 \end{pmatrix}.$$

Proof. For the proof, we first assume A and B in the form of Proposition 1.3. Then we get from its corollary

$$a_{14} = \delta^{1/2}/\lambda_1\lambda_4 = (\lambda_1\lambda_2\lambda_3\lambda_4)^{1/4}D, \qquad a_{23} = (a_{34}a_{13}\lambda_2 - \delta^{1/2})/\lambda_2\lambda_3,$$

$$a_{34} = \frac{\lambda_3}{a_{14}}a_{24} = \frac{\lambda_1\lambda_3\lambda_4}{(\lambda_1\lambda_2\lambda_3\lambda_4)^{3/4}}a_{24}, \qquad a_{13} = \frac{(\lambda_1\lambda_2\lambda_3\lambda_4)^{3/4}}{\lambda_1\lambda_2\lambda_4}a_{12},$$

where the last 2 equalities follow from $(BA)_{24} = 0$ and $(BA)_{42} = 0$. It follows from the equations above that $a_{34} = 0$ if and only if $a_{24} = 0$, and in this case, also $b_{21} = 0 = b_{31}$, by Prop. 1.3(b). But then $span\{w_2, w_3\}$ would be a B_3-submodule, contradicting simplicity of V. By Proposition 1.3,(e) we can choose $a_{24} = \lambda_4$, from which one deduces

$$a_{34} = \frac{\lambda_1\lambda_3\lambda_4^2}{(\lambda_1\lambda_2\lambda_3\lambda_4)^{3/4}}$$

We get from $(BA)_{34} = 0$, using the substitution in Prop. 1.3(b) that

$$a_{23} = (\lambda_1\lambda_2\lambda_3\lambda_4)^{1/4}(D+1).$$

Similarly, we obtain from $(BA)_{44} = 0$, using Prop. 1.3(b) and the results so far

$$a_{13} = \lambda_4^{-1}\lambda_2\lambda_3(D+1+D^{-1}), \quad a_{12} = \frac{\lambda_1\lambda_2^2\lambda_3}{(\lambda_1\lambda_2\lambda_3\lambda_4)^{3/4}}(D+1+D^{-1}).$$

To get A and B into the form as stated, it suffices to conjugate the matrices via the diagonal matrix $\mathrm{diag}(\lambda_4/a_{i4})_i$. This shows that there exist at most 2 representations of B_3 with prescribed eigenvalues λ_i for A, up to conjugation, depending on the choice of the square root in the expression for D.

Observe that we used the equations $(BA)_{ij} = 0$ for $j = 4$ and $i > 1$ for the computation of matrix entries above. It is easy to check $(BA)_{ij} = 0$ for the remaining entries for which $i + j > 5$. Hence condition (a) of Proposition 1.4 is satisfied, and condition (b) follows from Prop 1.3(d). Hence A and B satisfy the braid relations.

2.7. Proposition Let V be a simple 5-dimensional B_3-module. Then there exists a basis with respect to which A and B act as ordered triangular matrices. Moreover, the module is uniquely determined by the eigenvalues of A and the choice γ of a 5th root of $\det(A)$, up to equivalence.

Proof. We proceed as in the proof of Proposition 2.5, using the results of Section 1.3. By Corollary 1.3(a), we get

$$a_{15} = \frac{\delta^{1/2}}{\lambda_1\lambda_5} = \frac{\gamma^3}{\lambda_1\lambda_5}.$$

If $a_{35} = 0$, we can conclude from $(BA)_{35} = 0$ that $a_{34}a_{25} = a_{35}(\lambda_3 + a_{15}) = 0$. If $a_{34} = 0$, then also $b_{31} = 0 = b_{32}$ by Prop. 1.3(b). In this case, $\mathrm{span}\{w_1, w_2, w_4, w_5\}$ is a B_3-submodule. Hence we can assume $a_{34} \neq 0$ and $a_{25} = 0$. But then $a_{45} = (\lambda_4/a_{15})a_{25} = 0$ too by Corollary 1.3(b); and the matrix entries b_{21}, b_{31} and b_{41} are also equal to 0, by Prop. 1.3(b). Hence $\mathrm{span}\{w_1, w_5\}$ is an invariant subspace.

So we can assume $a_{35} \neq 0$. Assume $a_{45} = 0$; then also $a_{25} = (a_{15}/\lambda_4)a_{45} = 0$. We get from $(BA)_{45} = 0$ that $a_{35}a_{23} = \lambda_2 a_{45} + a_{24}a_{25} - a_{25}a_{15} = 0$. Since $a_{35} \neq 0$, $a_{23} = 0$. It follows that b_{21}, b_{41} and b_{43} are all equal to 0, by Prop 1.3(b). This would entail that $\mathrm{span}\{w_1, w_3, w_5\}$ is a B_3-submodule.

Hence we can assume that $a_{i5} \neq 0$ for $i = 1, 2, \ldots 5$. We can choose $a_{45} = \lambda_4$ and $a_{35} = a_{15}$, by Prop. 1.3(e). We now show that the equations in (1.3) completely determine the other entries of A and B. As this does not seem to be completely straightforward, we include the details for the interested reader. Using $(BA)_{25} = 0$ and $(BA)_{35} = 0$ we get

$$a_{25} = a_{15} = \frac{\gamma^3}{\lambda_1\lambda_5}, \qquad a_{34} = a_{15} + \lambda_3 = \frac{\gamma^3}{\lambda_1\lambda_5} + \lambda_3.$$

¿From $(BA)_{24} = -\frac{\delta^{1/2}}{\lambda_2} = -\frac{\gamma^3}{\lambda_2}$,

$$a_{24} = a_{14} - \frac{\gamma^3}{\lambda_2\lambda_4}.$$

Substituting this into $(BA)_{34} = 0$, we find

$$a_{14} = \frac{a_{34}(\lambda_2\lambda_3\lambda_4 + \gamma^3)}{(a_{34} - a_{35})\lambda_2\lambda_4} = \left(\frac{\lambda_2\lambda_4}{\gamma^2} + 1\right)\left(\lambda_3 + \frac{\gamma^3}{\lambda_2\lambda_4}\right),$$

and hence

$$a_{24} = \frac{\gamma^3}{\lambda_1\lambda_5} + \lambda_3 + \gamma.$$

Using $(BA)_{52} = 0$ we obtain

$$a_{12} = \frac{\lambda_2 a_{14}}{a_{15}} = \left(1 + \frac{\gamma^2}{\lambda_2\lambda_4}\right)\left(\lambda_2 + \frac{\gamma^3}{\lambda_3\lambda_4}\right).$$

Now $(BA)_{45} = 0$ gives us

$$a_{23} = a_{24} - a_{15} + \frac{\lambda_2\lambda_4}{a_{15}} = \gamma + \lambda_3 + \frac{\gamma^2}{\lambda_3}.$$

Finally, using $(BA)_{55}=0$, we get

$$a_{13}=\left(\frac{\gamma^2}{\lambda_3}+\lambda_3+\gamma\right)\left(1+\frac{\lambda_1\lambda_5}{\gamma^2}\right).$$

This shows that all entries of A and B are uniquely determined by the eigenvalues λ_i of A and by a choice of a 5^{th} root of $\det(A)$. It is now a straightforward computation to check the remaining conditions of Prop. 1.4 to prove that the matrices A and B do indeed define a representation of B_3. Alternatively, the existence question might be more easily settled using some of the methods discussed in Section 2.11.

2.8 Define for $1\le i\le d$ the polynomials $P_r^{(d)}(x)=\prod_{i\neq r}(x-\lambda_i)$, where $1\le i\le d$ with $i\neq r$. Observe that these polynomials divide the characteristic polynomial of A. It follows from the corollary of Prop. 2.2 that $P_r^{(d)}(A)$ is a nonzero rank 1 matrix; if the d eigenvalues of A are mutually distinct, it is a multiple of the projection onto the eigenspace of λ_r, with kernel being the direct sum of the eigenspaces of the other eigenvalues.

Now observe that for our braid representations, A and B are matrices with coefficients in the ring R of Laurent polynomials in the λ_i's and γ, a d-th root of $\det(A)$ for $d=4,5$. As also the matrix $P_r^{(d)}(A)P_s^{(d)}(A)P_r^{(d)}(A)$ is a multiple of $P_r^{(d)}$, we obtain Laurent polynomials $Q_{rs}^{(d)}$ in R by

$$P_r^{(d)}(A)P_s^{(d)}(B)P_r^{(d)}(A)=Q_{rs}^{(d)}P_r^{(d)}(A).$$

Proposition (a) $P_1^{(d)}(B)P_d^{(d)}(A)=Q_{1d}^{(d)}E_{dd}$, where E_{dd} is the matrix with a 1 in the (dd)-entry and zeroes everywhere else.

(b) If $Q_{ij}^{(d)}=0$, then $P_i^{(d)}(B)P_j^{(d)}(A)=0$.

(c) The matrix $(\prod_{r=i}^{d}(A-\lambda_r))$ has nonzero entries in its last column at most in the 1-st until $i-1$-st row.

(d) The Laurent polynomials $Q_{rs}^{(d)}$ are given by

$$Q_{rs}^{(2)}=-\lambda_r^2+\lambda_r\lambda_s-\lambda_s^2,\quad Q_{rs}^{(3)}=(\lambda_r^2+\lambda_s\lambda_k)(\lambda_s^2+\lambda_r\lambda_k),$$

$$Q_{rs}^{(4)}=-\gamma^{-2}(\lambda_r^2+\gamma^2)(\lambda_s^2+\gamma^2)(\gamma^2+\lambda_r\lambda_k+\lambda_s\lambda_l)(\gamma^2+\lambda_r\lambda_l+\lambda_s\lambda_k),$$

where $\{r,s,k,l\}=\{1,2,3,4\}$.

$$Q_{rs}^{(5)}=\gamma^{-8}(\gamma^2+\lambda_r\gamma+\lambda_r^2)(\gamma^2+\lambda_s\gamma+\lambda_s^2)\prod_{k\neq r,s}(\gamma^2+\lambda_r\lambda_k)(\gamma^2+\lambda_s\lambda_k).$$

Proof. The statements are shown by straightforward computations. We give some details for the interested reader. For statement (a), observe that $P_1^{(d)}(B)$ is nonzero only in the last row, where it coincides with the right-eigenvector of B for the eigenvalue λ_1, and that $P_d^{(d)}(A)$ is nonzero only in the last column, where it coincides with the left eigenvector of A for the eigenvalue λ_d. Using the triangular form of the matrices, and the fact that the nonzero diagonal entries are $P_1^{(d)}(\lambda_1)$ and $P_d^{(d)}(\lambda_d)$, respectively, these matrices can be computed easily. It is also obvious that $P_1^{(d)}(B)P_d^{(d)}(A)$ is a multiple of E_{dd}, and, multiplying it by $P_d^{(d)}(A)$ from the left, that this multiple is equal to $Q_{1d}^{(d)}$.

Statement (b) follows from (a) and the fact that we obtain matrices in ordered triangular form independent of the labeling, Prop. 2.4. Statement (d) follows similarly from (a). Statement (c) is straightforward.

2.9. Main theorem Let K be an algebraically closed field. (a) Any simple B_3 module is uniquely determined by the eigenvalues of A, up to a choice of a square root γ^2 of $\det(A)$ (for $d = 4$) resp. a 5-th root γ of $\det(A)$ (for $d = 5$).

(b) There exists a simple B_3 module V of K-dimension $d \leq 5$ if and only if the eigenvalues λ_i of A and the quantities γ^2 (for $d = 4$) and γ (for $d = 5$), as defined in (a), satisfy $Q_{rs}^{(d)} \neq 0$ for $r \neq s$ and $1 \leq r, s \leq d$, with $Q_{rs}^{(d)}$ as defined in 2.8. The eigenvalues λ_i need not be mutually distinct for this statement.

Proof. Let V be a simple B_3-module. By Propositions 2.5-7, we can assume a basis for V such that A and B act via matrices as described there. This shows part (a).

It remains to be shown for which values of the parameters the representations given there are simple. Let us assume $Q_{ij}^{(d)} = 0$ for some $i \neq j$, $1 \leq i, j \leq d$. If V is simple, we can assume $P_i^{(d)}(A) \neq 0$, by the corollary of Prop. 2.2. Hence there exists a vector v for which $e_i = P_i^{(d)}(A)v$ is an eigenvector of A. As $Q_{ji}^{(d)} = 0$, also $P_j^{(d)}(B)P_i^{(d)}(A) = 0$, by Prop. 2.8(b). Obviously, e_i is in the subspace spanned by $\{B^r e_i, 0 \leq r \leq d-2\} \cup \{b_i\}$, which is B-invariant; it is a proper subspace of V as the minimal polynomial of the restriction of B to it divides $P_j^{(d)}$. We obtain the existence of a non-trivial B_3-submodule from Prop. 2.2, i.e. V can not be simple.

On the other hand, assume that $Q_{ij}^{(d)} \neq 0$ for all $1 \leq i \neq j \leq d$. Let W be a non-zero B_3-submodule of V. Then it contains at least one eigenvector of A, say e_i. As $P_i^{(d)}(A) \neq 0$ by Prop. 2.8(a), we can assume this eigenvector to be of the form $e_i = P_i^{(d)}(A)v$ for some $v \in V$, also if A is not diagonalizable. Let $j \neq i$. Then $= ABAP_j^{(d)}(B)P_i^{(d)}(A)v$ is an eigenvector of A with eigenvalue λ_j, provided it is nonzero. Using $(ABA)^{-1}P_i(B)ABA = P_i(A)$, one easily checks that $P_i^{(d)}(B)e_j = Q_{ij}^{(d)}(ABA)e_i$, which is nonzero by our assumptions. Hence W contains eigenvectors for each eigenvalue of A.

Case 1: Assume A has at least 2 distinct eigenvalues, which we label λ_1 and λ_2. By triangularity of A, their eigenvectors can be chosen in the form $(1, 0, ...)^t$ and $(*, 1, 0, ...)^t$, which are both in W. It is easy to see from this that e_1, e_2, $b_1 = ABAe_1$ and $b_2 = ABAe_2$ together with Ab_1 (which is the last column of A) span V, which therefore is equal to W.

Case 2: Assume A and B has only one eigenvalue, say λ. Then $P_1^{(d)}(A)P_d^{(d)}(B) = (A-\lambda)^{d-1}(B-\lambda)^{d-1} = Q_{1d}E_{dd}$. As $Q_{1d} \neq 0$, the eigenspaces of λ for both A and B are 1-dimensional. If W is a non-zero B_3-submodule, it therefore must contain the eigenvector w_d of B. The set of vectors $S = \{(A-\lambda)^i w_d, i = 0, 1, \ ... \ d-1\}$ is in W. As $(A-\lambda)^{d-1} \neq 0$, so is its last column $(A-\lambda)^{d-1}w_d$, by triangularity of A. Hence S is linearly independent. This finishes the proof.

Corollary The simple $SL(2,\mathbf{Z})$ modules and the simple $PSL(2,\mathbf{Z})$ modules are given by all simple B_3-modules for which $\delta^2=1$ (for $SL(2,\mathbf{Z})$) and for which $\delta=1$ (for $PSL(2,\mathbf{Z})$). Observe that $\delta=-(\lambda_1\lambda_2)^3$ for $d=2$ and $\delta=(\lambda_1\lambda_2\lambda_3)^2$ for $d=3$.

2.10 Parameter spaces for B_3 and $PSL(2,\mathbf{Z})$ We define ideals $I_d \subset K[\lambda_1,\ \dots\ \lambda_d]$ (for $d\le 3$) and $I_d\subset K[\lambda_1,\ \dots\ \lambda_d,\gamma]$ (for $d=4,5$) as follows:

$I_2=\langle \lambda_1^2-\lambda_1\lambda_2+\lambda_2^2\rangle$,

$I_3=\langle \lambda_i^2+\lambda_r\lambda_s,\ \{i,r,s\}=\{1,2,3\}\rangle$,

$I_4=\langle \{\lambda_i^2+\gamma^2, i=1,2,3,4\}\ \cup\ \{\gamma^2+\lambda_i\lambda_j+\lambda_r\lambda_2,\ \{i,j,r,s\}=\{1,2,3,4\}\}\ \cup\ \{\gamma^4=\lambda_1\lambda_2\lambda_3\lambda_4\}\ \rangle$,

$I_5=\langle \{\gamma^2+\gamma\lambda_i+\lambda_i^2,\ i=1,2,3,4\}\cup\{\gamma^2+\lambda_i\lambda_j,\ 1\le i<j\le 5\}\cup\{\gamma^5=\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\}\rangle$.

Let N_d be the zero set of I_d in K^d (resp. in K^{d+1} for $d>3$). Observe that N_d is invariant under the action of the symmetric group, acting via permuting the coordinates $\lambda_i,\ 1\le i\le 5$ in K^d. The main theorem can now be reformulated as follows:

Main Theorem′ There exists a 1-1 correspondence between equivalence classes of simple B_3 modules of dimension $d\le 5$ and the S_d orbits in $K^d\backslash N_d$.

2.11 Remarks 1. After this research was completed we became aware of a number of related approaches and results. We learned from P. Deligne that results similar to ours are obtained in N. Katz's work [K] on rigid local systems by a more general and less elementary method. In particular, he sketched to us how one can obtain the classification of 5-dimensional braid representations from Katz's work (see also Remark 4 below).

2. We also learned of another approach using quiver theory (see [Ws] and, for a more general approach, [S]). The following result appears explicitly in a preprint by Westbury:

Recall that in the $PSL(2,\mathbf{Z})$-quotient of B_3 the elements ABA and AB have order 2 and 3 respectively. Let n_1 and n_2 be the dimensions of the eigenspaces of ABA, and let m_1, m_2 and m_3 be the dimensions of the eigenspaces of AB. Then there exists an indecomposable representation of $PSL(2,\mathbf{Z})$ if and only if $n_i\ge m_j$ for $i=1,2$ and $j=1,2,3$. Moreover, the parameter space for representations of this type has dimension $d^2-n_1^2-n_2^2-m_1^2-m_2^2-m_3^2+1$. This result is in accordance with our findings for $d\le 5$, but does not say anything about when the representations are simple (e.g. for $d=4$ there also exist indecomposable but not simple representations with AB only having 2 eigenvalues).

3. We do not expect our methods to work for dimension ≥ 6 without significant changes. For one, we can not expect ordered triangular forms for A and B: Indeed, if this were the case, $Tr(ABA)=0$ for any simple 6-dimensional representation of B_3, by Prop. 1.3(a), i.e. $n_1=n_2=3$. To get a counterexample, it suffices to find 6×6 matrices of order 2 and 3 which generate the whole ring of 6×6 matrices and such that $n_1=4$ and $n_2=2$.

Similarly, the result quoted under 2 also shows that we will have more parameters than the eigenvalues for $d\ge 6$ and suitable values of m_1,m_2,m_3,n_1,n_2.

4. There exists a beautiful and simple argument which considerably narrows down

which polynomials can occur in the simplicity statement of Theorem 2.9, and which also explains the nature of the factors in the polynomials $Q_{ij}^{(d)}$ to some extent. We were told this argument by P. Deligne:

Let V be a d-dimensional B_3 module on which the central element $\zeta = (\sigma_1\sigma_2)^3$ acts as a scalar δ. Assume that V has an r-dimensional B_3-submodule W on which σ_1 acts with eigenvalues $\lambda_1, \ldots \lambda_r$. Comparing determinants of the restriction of ζ to W, we get that

$$(\lambda_1 \ldots \lambda_d)^{6r} = (\lambda_1 \ldots \lambda_r)^{6d}. \qquad (*)$$

Hence whenever ζ acts as a scalar and $(*)$ is NOT satisfied, the representation has to be simple. Unfortunately, this argument produces a sufficient but not a necessary condition for V being simple.

5. A result which could be related to our findings is Coxeter's classification of finite quotients of B_3 defined by the additional relation $\sigma_i^p = 1$, for $i = 1, 2$: the additional relation defines a finite quotient if and only if $p \leq 5$ (see [C]).

3. An Application to Tensor Categories

3.1 We shall use the results of the previous section to compute categorical dimensions for certain simple objects in braided tensor categories related to exceptional Lie groups. We will not give much background information about tensor categories here (see e.g. [DM], [JS],[KW] or [T]), as the application itself is rather elementary. The reader not familiar with the categorical language should think of the categories as representation categories of quantum groups.

Let $\mathcal{C}$ be a semisimple ribbon tensor category with trivial object $\mathbb{1}$ over an algebraically closed field K (see [T] for precise definitions); this means, in particular, that $End(X)$ is a semisimple K-algebra for any given object X in $\mathcal{C}$, and the homomorphisms between 2 objects in $\mathcal{C}$ form a vector space over K.

In the following, let Z be a simple selfdual object in $\mathcal{C}$; this means that the trivial object $\mathbb{1}$ appears with multiplicity one as a direct summand in $Z^{\otimes 2}$. Let $p \in \mathrm{End}(Z^{\otimes 2})$ be the projection onto $\mathbb{1}$, and let $p_1 = p \otimes 1$ and $p_2 = 1 \otimes p$ be elements in $\mathrm{End}(Z^{\otimes 3})$. In the following we will always assume the following (rigidity) condition

$$p_2 p_1 p_2 \neq 0 \qquad (3.1)$$

As $Z \otimes \mathbb{1}$ is canonically isomorphic to the simple object Z, $p_2(a \otimes 1)p_2$ is equal to a scalar multiple of p_2 for any $a \in \mathrm{End}(Z^{\otimes 2})$. Let Y be a direct summand in $Z^{\otimes 2}$, with respect to a chosen direct sum decomposition of $Z^{\otimes 2}$. This decomposition defines a projection p_Y onto Y. We define the categorical dimension $\dim Y$ by

$$(\dim Y)p_2 = (\dim Z)^2 p_2(p_Y \otimes 1)p_2. \qquad (3.2)$$

This definition depends on a choice of $\dim Z$. In the following we will always require $\dim\ \mathbb{1} = 1$. Taking for p_Y the projection p_1, it follows from (3.1) and (3.2) that $\dim Z$ is uniquely determined up to a sign; $\dim Y$ is independent of the choice of the sign. So, in particular, if $Z^{\otimes 2}$ contains a subobject isomorphic to Z, $\dim Z$ is uniquely determined by (3.2). It is not hard to check that for subobjects of $Z^{\otimes 2}$ this definition coincides with the usual definition of categorical trace as e.g. for ribbon tensor categories (see [T]).

3.2 Assume that $\mathcal{C}$ allows a braiding (see e.g. [JS], [T]). For the purpose of this article, it suffices to know that this entails that there exists a canonical endomorphism $c \in \mathrm{End}(Z^{\otimes 2})$ such that $c_1c_2c_1 = c_2c_1c_2$, with c_i elements in $\mathrm{End}(Z^{\otimes 3})$ defined as before the p_i. We have the following simple

Lemma Assume that $Z^{\otimes 2}$ decomposes as a direct sum $\bigoplus_i Y_i$ of d mutually nonisomorphic simple objects Y_i, each of which has non-zero dimension. Moreover, we assume that c acts on Y_i via the scalar λ_i, and $\lambda_i \neq \lambda_j$ for $i \neq j$.

Then $\mathrm{Hom}(Z, Z^{\otimes 3})$ is a simple d-dimensional B_3-module, with the action defined via $\sigma_j f = c_j \circ f$ for all $f \in \mathrm{Hom}(Z, Z^{\otimes 3})$ and $j = 1, 2$; moreover, each eigenvalue of c_j has multiplicity 1.

Proof. Let $p^{(i)} \in \mathrm{End}(Z^{\otimes 2})$ be the projection onto Y_i, with $p = p^{(1)}$ being the projection onto $\mathbb{1} \subset Z^{\otimes 2}$ and let $\iota : \mathbb{1} \to Z^{\otimes 2}$ be a nonzero homomorphism. It follows from rigidity that $(p^{(i)} \otimes 1) \circ (1 \otimes \iota)$ is a nonzero homomorphism and that $\dim \mathrm{Hom}(Z, Z^{\otimes 3}) = \dim \mathrm{End}(Z^{\otimes 2}) = d$; it can also be checked explicitly for the examples below. Hence $V = \mathrm{Hom}(Z, Z^{\otimes 3})$ is a d-dimensional vector space on which both c_1 and c_2 act via concatenation of morphisms. Moreover, it has a basis of eigenvectors $(p^{(i)} \otimes 1) \circ (1 \otimes \iota)$ of c_1. If this representation of B_3 were not simple, fix a composition series of V, and pick an eigenprojection $p^{(i)}$ of c such that the eigenprojections $p^{(i)} \otimes 1$ and $p_1 = p \otimes 1$ of c_1 act nonzero on different simple factors. As p_1 is conjugate to $p_2 = 1 \otimes p$ (as c_1 and c_2 are conjugate), we can conclude $p_2(p^{(i)} \otimes 1)p_2 = 0$, which would contradict the assumption about nonzero dimensions.

Corollary Using the notations of the Lemma, with $Y_1 \cong \mathbb{1}$, we have

$$\dim Y_i = \frac{Q_{1i}^{(d)} (\dim Z)^2}{P_1^{(d)}(\lambda_1) P_i^{(d)}(\lambda_i)},$$

where Q_{1i}, $P_1^{(d)}(\lambda_1)$ and $P_i^{(d)}(\lambda_i)$ are as in Section 2.8.

Proof. Observe that the projection onto the eigenspace of A with eigenvalue λ is given by $P_i^{(d)}(A)/P_i^{(d)}(\lambda_i)$ The claim now follows from (3.2) and Prop. 2.8(a).

3.3 Let us first consider the case with $\mathcal{C}$ a braided tensor category whose Grothendieck semiring is isomorphic to the semiring of the representation category of the Lie group G with G being an orthogonal group $O(N)$ or a symplectic group $Sp(N)$. We take as object Z the vector representation of G. It is well-known that in this case $Z \otimes Z \cong \mathbb{1} \oplus X \oplus Y$, with X and Y simple objects corresponding to the antisymmetrization and a subrepresentation of the symmetrization (traceless tensors) of the 2nd tensor power of the vector representation. It can be shown that in this case (only using the assumptions of braiding, and the given Grothendieck semi-ring), the eigenvalues are of the form $\alpha q, -\alpha q^{-1}, \alpha r^{-1}$, with α a 4-th root of unity and $q, r \in K$. Moreover, r is $\pm$ a power of q, depending on the given category (see below). Using the notations $[n] = q^n - q^{-n}$ and $[\lambda + n] = rq^n - r^{-1}q^{-n}$, it follows from Corollary 3.2 that

$$\dim Z = \alpha^2(\frac{[\lambda]}{[1]} + 1) = \alpha^2(\frac{r - r^{-1}}{q - q^{-1}} + 1),$$

and

$$\dim X = \frac{[\lambda - 1] + [2]}{[2]} \frac{[\lambda]}{[1]}, \quad \dim Y = \frac{[\lambda + 1] + [2]}{[2]} \frac{[\lambda]}{[1]}.$$

This method can be extended to define similar functions also for objects in higher tensor powers of Z, which are labeled by Young diagrams (see [Wn, Theorem 5.5]). The q-dimensions for $G = O(N)$ are obtained by setting $\lambda = N - 1$, for $G = Sp(N)$ they are obtained by setting $\lambda = -N-1$. More generally, it is possible to reconstruct such categories similarly as it was done for categories of type A in [KW]; however, in this case we would need the assumption of the category being braided. So in this comparatively simple case knowledge about braid representations allows us to reconstruct the tensor category.

3.4 The discussion in 3.3 can be carried over to exceptional Lie groups to some (so far rather limited) extent. We use the notations $[n] = s^{n/2} - s^{-n/2}$ and $[\lambda+n] = t^{1/2}s^{n/2} - t^{-1/2}s^{-n/2}$ for $n \in \mathbf{N}$ and λ a formal variable; in our formulas, the result will be independent of the choice of square roots of s and t. It can be shown that for an exceptional Lie algebra $\mathfrak{g}$, the 2nd tensor product $\mathfrak{g}^{\otimes 2}$ of its adjoint representation decomposes as a direct sum $\mathbb{1} \oplus \mathfrak{g} \oplus X_2 \oplus Y_2 \oplus Y_2^*$. Here we use the same notations as in [De]. We are going to use the following results about quantum groups, which are essentially due to Drinfeld [Dr2] (for a description of the relationship between operators c (braiding operators) and quantum Casimir (= twist) also see e.g. [T]); here a_Y denotes the scalar via which the Casimir acts on a simple $\mathfrak{g}$-module Y:

(i) the braiding operator c acts on the simple summand $Y \subset \mathfrak{g}^{\otimes 2}$ via the scalar $\pm q^{a_Y - 2a_{\mathfrak{g}}}$, with the sign depending on whether Y is in the symmetrization or antisymmetrization of $\mathfrak{g}^{\otimes 2}$,

(ii) the operator $(c_1c_2)^3$ acts on the simple summand $K \subset \mathfrak{g}^{\otimes 3}$ via the scalar $q^{2a_K - 6a_{\mathfrak{g}}}$.

Now setting $q^{2a_{\mathfrak{g}}} = s^{-3}$ and $q^{a_{Y_2} - 2a_{\mathfrak{g}}} = t$, and using the formulas for the action of the Casimir in [De], one sees easily that the eigenvalues of c (acting on $\mathfrak{g}^{\otimes 2}$) are s^6 (for $\mathbb{1}$), $-s^3$ (for $\mathfrak{g}$), -1 (for X_2), t (for Y_2) and st^{-1} (for Y_2^*). It follows from this that $\det_V(c_1) = s^{10}$, where $\det_V(c_1)$ is the determinant of the linear operator via which c_1 acts on V.

In order to completely determine the 5-dimensional representation of B_3 on $V = \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}^{\otimes 3})$, we only need to compute the scalar δ by which the central element $(c_1c_2)^3$ acts on V. This is equivalent to determining the 5-th root γ of $\det_V(c_1)$ given by $\delta = \gamma^6 = \det_V(c_1)\gamma = s^{10}\gamma$. By (ii), we get $\delta = q^{2a_{\mathfrak{g}} - 6a_{\mathfrak{g}}} = s^{12}$. Combining the last 2 formulas, we get $\gamma = s^2$. One can now check, using Corollary 3.2, that

$$\dim \mathfrak{g} = \frac{[4][\lambda - 6][\lambda + 5]}{[2][\lambda - 1][\lambda]},$$

$$\dim X_2 = \frac{[5][\lambda-6][\lambda+5][\lambda-4][\lambda+3][2\lambda+4][2\lambda-6]}{[1][\lambda][\lambda-1][\lambda+2][\lambda-3][2\lambda][2\lambda-2]},$$

$$\dim Y_2 = \frac{[6][5][4][\lambda+5][\lambda-4][3\lambda-6]}{[2][\lambda-1][\lambda][2\lambda][2\lambda-1][\lambda-2]},$$

$$\dim Y_2^* = \frac{[6]][5][4][\lambda-6][2\lambda+6][3\lambda+3][\lambda+2]}{[2][\lambda-1][\lambda][2\lambda-2][2\lambda-1][2\lambda+4][\lambda+1]}.$$

These rational functions are s-deformations of formulas given in [De] for the classical case.

3.5 The computations above in the orthogonal case were used by Toledano Laredo in [TL] in the context of fusion of representations of loop groups, based on unpublished notes of the second named author. We have been informed by Antony Wassermann that he uses our results for 5-dimensional braid representations in connection with fusion of representations of loop groups corresponding to exceptional Lie groups.

References

[BLM] J.S Birman, D. D. Long and J. A. Moody Finite-dimensional representations of Artin's braid group, Contemp. Math., 169, Amer. Math. Soc., Providence, RI, 1994, 123–132.

[C] H.S.M. Coxeter, Factor groups of the braid group, Proceedings of the Fourth Can. Math. Congress, Banff 1957, University of Toronto Press (1959), 95-122

[De] P. Deligne, La série exceptionnelle de groupes de Lie. C. R. Acad. Sci. Paris Sr. I Math. 322 (1996), no. 4, 321–326.

[DM] P. Deligne and J. Milne, Tannakian categories, Lecture Notes in Mathematics, vol 900, Springer Verlag

[Dr1] V. Drinfeld, Quantum groups, Proceedings for the ICM Berkeley 1986 798-820.

[Dr2] V. Drinfeld, On almost cocommutative Hopf algebras, Leningrad Math. J. 1 (1990) 321-243.

[JS] A. Joyal and R. Street, The geometry of tensor calculus, Adv. Math. 88 (1991)

[K] N. Katz, Rigid local systems, Annals of Mathematics Studies, 139, Princeton University Press

[KW] D. Kazhdan and H. Wenzl, Reconstructing monoidal categories, Advances in Soviet Math. Vol 16 Part 2 (1993) 111-136

[S] A. Schofield, General representations of quivers. Proc. London Math. Soc. (3) 65 (1992), no. 1, 46–64.

[TL] V. Toledano Laredo, Fusion of positive energy representations of LSpin_{2n}, Thesis, University of Cambridge, 1997

[T] V.G. Turaev, Quantum Invariants of Knots and 3-Manifolds, de Gruyter (1994)

[V] P. Vogel, Algebraic structures on modules of diagrams, preprint

[Wa] A. Wassermann, Operator algebras and conformal field theory III, Invent. Math. (1998)

[Wn] H. Wenzl, Quantum groups and subfactors of type B, C and D, Comm. Math. Phys. 133 (1990) 383-432

[Ws] B. Westbury, On the character varieties of the modular group, preprint, University of Nottingham, 1995

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