UNIVERSITY OF CALIFORNIA, SAN DIEGO

Braid Representations and Tensor Categories

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by

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The dissertation of Imre Tuba is approved, and it is acceptable in quality and form for publication on microfilm:

Chair

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To Kip

Minden, minden hogy elmarad
S hogy elhagyunk mindent, mindent
Előbb-utóbb.

(Ady Endre)

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The text of Chapter 2 is similar to the presentation in [11], while Chapter 3 closely follows [10].

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PUBLICATIONS

(with Hans Wenzl) Representations of the Braid Group B_3 and of $SL(2,\mathbb{Z})$. To appear *Pacific J. Math.* Preprint posted at `http://euclid.ucsd.edu/~ituba` and `http://xxx.lanl.gov`.

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ABSTRACT OF THE DISSERTATION

Braid Representations and Tensor Categories

by

Imre Tuba
Doctor of Philosophy in Mathematics

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Professor Hans Wenzl, Chair

We classify all simple representations of the braid group B_3 with dimension $d \leq 5$ over any algebraically closed field. In particular, we prove that a simple d-dimensional representation $\rho : B_3 \to GL(V)$ is determined up to isomorphism by the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_d$ of the image of the generators σ_1 and σ_2 for $d = 2, 3$ and a choice of a $\delta = \sqrt{\det \rho(\sigma_1)}$ for $d = 4$ or a choice of $\delta = \sqrt[5]{\det \rho(\sigma_1)}$ for $d = 5$. We also showed that such representations exist whenever the eigenvalues and δ are not zeros of certain explicitly given rational functions $Q_{ij}^{(d)}$. In this case, we construct the matrices via which the generators act on V.

We go on to give a necessary and sufficient condition for the unitarizability of simple representations of B_3 on complex vector spaces of dimension $d \leq 5$. We show that a simple representation $\rho : B_3 \to \mathrm{GL}(V)$ (for $\dim V \leq 5$) is unitarizable if and only if the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_d$ of $\rho(\sigma_1)$ are distinct, satisfy $|\lambda_i| = 1$ and $\mu_{i1}^{(d)} > 0$ for $2 \leq i \leq d$, where the $\mu_{i1}^{(d)}$ are functions of the eigenvalues, related to $Q_{ij}^{(d)}$.

Finally, we describe how these results can be used to compute categorical dimensions of objects in braided tensor categories and give an example of such a computation.

Chapter 1

Introduction

Braids were first studied by Emil Artin in 1926. He later introduced the braid group B_n in [2] arising from his geometric construction of braids. We will give a purely algebraic definition instead. B_n is generated by the $n-1$ generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ subject to the braid relations:

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i \qquad \text{if } |i-j| \neq 1 \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}. \end{aligned}$$

B_n can be visualized as the braiding on n strings by identifying σ_i with crossing the i-th string over the $i+1$-st string.

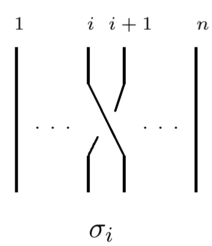
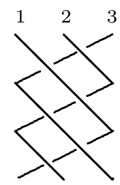


Figure 1.1: A generator of B_n

Note that the braid relations imply $(\sigma_i \sigma_{i+1}) \, \sigma_i \, (\sigma_i \sigma_{i+1})^{-1} = \sigma_{i+1}$, that is all of the generators are in the same conjugacy class. Furthermore, it is easy to see that the full twist $\Delta_n^2 = (\sigma_{n-1} \dots \sigma_1)^n$ is central. It is not obvious but Δ_n^2 actually generates the center (see p. 28 in [3] for a proof).



Δ_3^2

Figure 1.2: The full twist in B_3

It is worth noting that $B_3/Z(B_3) \cong \mathrm{PSL}_2(\mathbb{Z})$ where $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/Z(\mathrm{SL}_2(\mathbb{Z}))$ via the map

$$\sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Hence any representations of B_3 easily give rise to representations of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{PSL}_2(\mathbb{Z})$ and vice versa. $PSL_2(\mathbb{Z})$ is also isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_3$.

Representations of the braid group appear in many places in topology and algebra. In particular, my interest in braid representations derives from their applications to braided tensor categories.

Chapter 2

Simple Representations of Dimension $d \leq 5$

2.1 Introduction

In this chapter, we will characterize all simple representations of B_3 of dimension $d \leq 5$. We will mostly follow the argument presented in [11] with some variations.

Suppose $\rho : B_3 \to \mathrm{GL}(V)$ is such a simple representation of B_3 on the d-dimensional vector space V over an algebraically closed field F of any characteristic. Denote the images of σ_1 and σ_2 by A and B. In general, we will talk about V as a B_3-module, where B_3 acts on V via ρ.

Then A and B are invertible $d \times d$ matrices with entries in F. They satisfy $ABA = BAB$ and

Proposition 2.1.

a) $B = (AB)\,A\,(AB)^{-1} = (ABA)\,A\,(ABA)^{-1}$ *and* $A = (BA)\,A\,(BA)^{-1} = (ABA)\,A\,(ABA)^{-1}$.

b) The eigenvalues of A and B are the same.

c) If $\{a_1, a_2, \ldots, a_d\}$ is a basis of eigenvectors of A, then $\{b_1, b_2, \ldots, b_d\}$ with $b_i = (ABA)\,a_i$ is a basis of eigenvectors of B.

d) $(ABA)^2 = (AB)^3 = \delta\, I$ *where* $\delta = \det(A)^{6/d}$. *Hence* $(ABA)^{-1} = \delta^{-1}(ABA)$.

e) The map $\rho' : B_3 \to \mathrm{GL}(V)$ defined by $\rho'(\sigma_1) = \delta^{-1/6}A$ and $\rho'(\sigma_2) = \delta^{-1/6}B$ is still a representation of B_3 for any choice of the sixth root. It has the property $\det(\rho'(\sigma_i)) = 1$ for $i = 1, 2$.

Proof: (a) follows from the corresponding relations in B_3. (b) and (c) follow from (a). Note that A and B generate all of $M_d(F)$ as ρ is assumed to be a simple representation. We know $(\sigma_1\sigma_2)^3$ is in the center of B_3, so $(AB)^3$ is in the center of $M_d(F)$, hence it is a scalar matrix $\delta\, I$. Observe that $\delta^d = \det(\delta\, I) = \det(AB)^3 = \det(A)^6$ by (a). This proves (d). (e) is obvious. $\square$

2.2 A Particularly Nice Example

The key observation, motivated by the following example that will enable us to compute A and B is that we can assume them to be in a special form.

Example 2.2. Only in this example, we will index the basis starting with 0 and we will redefine $\bar{i} = d - i$. So let V be a $d+1$-dimensional vector space with $\{v_0, v_1, \ldots, v_d\}$ as a basis and $\lambda_i\lambda_{\bar{i}} = \gamma \neq 0$ for some fixed $\gamma \in F$ and $0 \le i \le d$.

$$A = \left(\binom{\bar{i}}{\bar{j}}\lambda_j\right)_{ij} = \begin{pmatrix} \lambda_0 & \binom{d}{d-1}\lambda_1 & \binom{d}{d-2}\lambda_2 & \cdots & \lambda_d \\ & \lambda_1 & \binom{d-1}{d-2}\lambda_2 & \cdots & \lambda_d \\ & & \lambda_2 & & \vdots \\ & & & \ddots & \vdots \\ & & & & \lambda_d \end{pmatrix}$$

and

$$B = \left((-1)^{i+j}\binom{i}{j}\lambda_{\bar{i}}\right)_{ij} = \begin{pmatrix} \lambda_d & & & & \\ -\lambda_{d-1} & \lambda_{d-1} & & & \\ \vdots & & \ddots & & \\ (-1)^{d-1}\lambda_1 & (-1)^d\binom{d-1}{1}\lambda_1 & \cdots & \lambda_1 & \\ (-1)^d\lambda_0 & (-1)^{d+1}\binom{d}{1}\lambda_0 & (-1)^{d+2}\binom{d}{2}\lambda_0 & \cdots & \lambda_0 \end{pmatrix}$$

satisfy $ABA = BAB$, and hence yield a representation of B_3.

Proof: By direct computation and Lemma 2.3,

$$(AB)_{ij} = \sum_{k=0}^{d} (-1)^{k+j} \binom{\bar{i}}{k} \lambda_k \binom{k}{j} \lambda_{\bar{k}} = (-1)^{d+j} \binom{i}{\bar{j}} \gamma.$$

This shows that AB is lower skew-diagonal, that is $(AB)_{ij} = 0$ for $i < \bar{j}$.

Let

$$S' = \begin{pmatrix} & & & \lambda_d \\ & & -\lambda_{d-1} & \\ & \lambda_{d-2} & & \\ & & & \\ (-1)^d \lambda_0 & & & \end{pmatrix}.$$

Note $S'^2 = (-1)^d \, \gamma \, I$, hence

$$S'^{-1} = (-1)^d \, \gamma^{-1} \, S = \begin{pmatrix} & & & (-1)^d \lambda_0^{-1} \\ & & & \\ & & \lambda_{d-2}^{-1} & \\ & -\lambda_{d-1}^{-1} & & \\ \lambda_d^{-1} & & & \end{pmatrix}.$$

By the above, $\lambda_{\bar{i}} \, (AB)_{i\bar{i}} = (-1)^i \, \gamma \, \lambda_{\bar{i}} = \gamma \, S'_{i\bar{i}}$. Let $S = \gamma S'$.

Also,

$$(SAS^{-1})_{ij} = S' \, A \, S'^{-1} = (-1)^{\bar{i}} \, \lambda_{\bar{i}} \, A_{\bar{i}\bar{j}} \, (-1)^{\bar{j}} \, \lambda_{\bar{j}}^{-1} =$$

$$(-1)^{i+j} \, \frac{\lambda_{\bar{i}}}{\lambda_{\bar{j}}} \binom{i}{j} \lambda_{\bar{j}} = B_{ij}$$

Thus A, B and S satisfy the conditions of Lemma 2.11. □

Lemma 2.3. *For* $0 \le i, j \le d$,

$$\sum_{k=0}^{d} (-1)^k \binom{d-i}{k} \binom{d-k}{j} = \binom{i}{d-j}.$$

Proof: Expand

$$(1+x)^i \; ((1+x)+y)^{n-i} = (1+x)^i \sum_{k=0}^{n-i} \binom{n-i}{k} y^k \, (1+x)^{n-i-k} =$$

$$\sum_{k=0}^{n-i} \binom{n-i}{k} y^k (1+x)^{n-k} = \sum_{k=0}^{n-i} \sum_{l=0}^{n-k} \binom{n-i}{k} \binom{n-k}{l} y^k x^l.$$

Now substitute $y = -1$:

$$(1+x)^i x^{n-i} = \sum_{k=0}^{n-i} \sum_{l=0}^{n-k} \binom{n-i}{k} \binom{n-k}{l} (-1)^k x^l.$$

Also

$$(1+x)^i x^{n-i} = \sum_{k=0}^{i} \binom{i}{k} x^{i-k} x^{n-i} = \sum_{k=0}^{i} \binom{i}{k} x^{n-k}.$$

Now equating coefficients of x^j results in the desired identity. $\square$

2.3 Preliminary Results

Definition 2.4. We will say that A and B are in *ordered triangular* form if A is upper triangular with the $\lambda_1, \lambda_2, \ldots, \lambda_d$ down its diagonal, while B is lower triangular with $\lambda_d, \lambda_{d-1}, \ldots, \lambda_1$ down its diagonal.

We will eventually prove that A and B can be assumed to be in ordered triangular form without any loss of generality. But first, we need a few auxiliary results.

Let $\lambda_1, \lambda_2, \ldots, \lambda_d$ be the (not necessarily distinct) eigenvalues of A with corresponding (generalized) eigenvectors $a_1, a_2, \ldots, a_d$. By Proposition 2.1, they are also the eigenvalues of B corresponding to the (generalized) eigenvectors $b_i = ABA\, a_i$.

Lemma 2.5.

a) $\rho(B_3) = \langle A,\, B\rangle = \langle A,\, AB\rangle = \langle B,\, AB\rangle = \langle A,\, ABA\rangle = \langle B,\, ABA\rangle = \langle AB,\, ABA\rangle$

b) If I is any subset of $\{1, 2, \ldots, d\}$, then $W = \operatorname{span}\{a_i \mid i \in I\} \cap \operatorname{span}\{b_i \mid i \in I\}$ is invariant under $\rho(B_3)$.

Proof: (a) is an obvious consequence of the analogous statements for B_3. For (b), note that W is invariant under $\langle A,\, B\rangle$. $\square$

Lemma 2.6. *Let V be a simple B_3-module of dimension $d \geq 3$ and $V_1 = \operatorname{span}\{a_1, \ldots, a_{d-1}\}$, $V_2 = V_1 \cap (ABA)\, V_1$ and $V_3 = V_1 \cap (AB)\, V_1 \cap (AB)^2\, V_1$. Then V_2 is ABA-invariant and V_3 is AB-invariant. Moreover $V_3 \subsetneq V_2 \subsetneq V_1 \subsetneq V$ and* $\operatorname{codim} V_1 = 1$, $\operatorname{codim} V_2 = 2$, *and* $\operatorname{codim} V_3 = 3$.

Proof: The invariance statements are obvious. Note that $(ABA)\,a_i = \lambda_i\,(AB)\,a_i$, so $(ABA)\,V_1 = (AB)\,V_1$. Hence $V_3 \subseteq V_1 \cap (AB)\,V_1 = V_2$.

Note that $\operatorname{codim} V_2 \leq 2$ as V_2 is the intersection of two subspaces of codimension 1. If $W = V_2$, then $W = \operatorname{span}\{b_1, \ldots, b_{d-1}\}$. But $\dim V_1 = d-1$, so V_1 would be a proper invariant subspace contradicting simplicity by Lemma 2.5. So $V_2 \subsetneq V_1$ and $\operatorname{codim} V_2 = 2$.

By the same logic, $\operatorname{codim} V_3 \leq 3$. If $V_2 = V_3$, then V_2 is a proper subspace invariant under $\langle ABA,\, AB\rangle = \rho(B_3)$, which contradicts simplicity. Hence $V_3 \subsetneq V_2$ and $\operatorname{codim} V_3 = 3$. □

Proposition 2.7. *If V is a simple B_3 module of dimension $d \leq 5$ and W is a proper subspace of V invariant under A or B then W cannot contain both a_i and $b_i = (ABA)\,a_i$.*

Proof: If W is a proper A-invariant subspace that contains a_i and b_i, then $(ABA)\,W$ is a proper B-invariant subspace that contains $a_i = \delta^{-1}\,ABA\,b_i$ and $b_i = ABA\,a_i$. So we may assume without loss of generality that W is B-invariant by replacing it with $(ABA)\,W$ if necessary.

Suppose W is B-invariant and does contain both a_i and b_i. We are free to reindex the eigenvalues if necessary, so we may assume without loss of generality that $i = 1$. Let $d' = \dim W$. Extend b_1 to a basis of generalized eigenvectors $\langle b_1, \ldots, b_{d'}\rangle$ of B on W, then extend to an eigenbasis of V. Let $a_i = (ABA)^{-1}\,b_i = \delta^{-1}(ABA)\,b_i$. Let $V_1 = \operatorname{span}\{b_1, \ldots, b_{d-1}\}$ and V_2, V_3 as in Lemma 2.6.

$a_1 \in W \subseteq V_1$. Hence $a_1 \in V_2 = V_1 \cap \operatorname{span}\{a_1, \ldots, a_{d-1}\}$. As V_2 is ABA-invariant, $b_1 = (ABA)\,a_1 \in V_2$. In particular $V_2 \neq 0$.

By Lemma 2.6, $\dim V_2 = d - 2$, which immediately leads to contradiction if $d = 2$.

If $d = 3$, then $\dim V_2 = 1$, so $V_2 = \operatorname{span}\{a_1\} = \operatorname{span}\{b_1\}$, which contradicts Lemma 2.5.

If $d = 4$, $\dim V_2 = 2$ and $\dim V_3 = 1$. If $a_1 \in V_3$, then $V_3 = \operatorname{span}\{a_1\}$, so V_3 is invariant under $\langle A, AB\rangle = \rho(B_3)$, which contradicts simplicity. So $b_1 \in V_2 = \operatorname{span}\{a_1\} + V_3$, that is $b_1 = \alpha a_1 + w$ for some $\alpha \in F$ and $w \in V_3$. Then

$$(AB)\,b_1 = AB(\alpha a_1 + w) = \alpha(ABA)\,A^{-1}a_1 + (AB)\,w = \lambda_1^{-1}\alpha b_1 + (AB)\,w \in V_2.$$

Hence V_2 is invariant under $\langle ABA,\, AB\rangle = \rho(B_3)$, which is again a contradiction.

If $d = 5$, the argument is similar. Now $\dim V_2 = 3$ and $\dim V_3 = 2$. If $a_1 \in V_3$, then $b_1 = (ABA)\, a_1 = \lambda_1 (AB)\, a_1 \in V_3$ too. Hence $V_3 = \operatorname{span}\{a_1,\, b_1\}$, so V_3 is invariant under $\langle AB,\, ABA\rangle = \rho(B_3)$ contradicting simplicity. Therefore $V_2 = \operatorname{span}\{a_1\} + V_3 = \operatorname{span}\{b_1\} + V_3$. Hence $b_1 = \alpha a_1 + w$ for some $\alpha \in F$ and $w \in V_3$, and

$$(AB)\, b_1 = AB(\alpha a_1 + w) = \alpha(ABA)\, A^{-1} a_1 + (AB)\, w = \lambda_1^{-1} \alpha b_1 + (AB)\, w \in V_2.$$

Thus $V_2 = \operatorname{span}\{b_1\} + V_3$ is a proper subspace invariant under $\langle ABA,\, AB\rangle = \rho(B_3)$.

The statement about an A-invariant subspace follows either by a symmetric argument, or simply by noting that $(ABA)\, W$ would then be B-invariant and would still contain a_i and b_i. □

Lemma 2.8. *If A is diagonalizable, then its eigenvalues are distinct.*

Proof: Let A be diagonalizable. If all of the eigenvalues are the same, then $A = \lambda I$ and hence $B = \lambda I$. But then any subspace is invariant, so V must be 1-dimensional and the statement holds.

Suppose not all eigenvalues are distinct. So we may assume without loss of generality, that $\lambda_1 = \lambda_2$ and $\lambda_d \neq \lambda_1$. Let b be an eigenvector of B that corresponds to λ_d. Let $W = \operatorname{span}\left\{A^i\, b \mid i = 0, \ldots, d-2\right\}$. Since the minimal polynomial has at most degree $d-1$, W is A-invariant. Note that $W = \operatorname{span}\left\{(A - \lambda_1)^i\, b \mid i = 0, \ldots, d-2\right\}$. Let $w_i = (A - \lambda_1)^i\, b$ and n such that $w_n \neq 0$ but $w_{n+1} = 0$.

Then it is easy to see that $\{w_0, w_1, \ldots, w_n\}$ is a basis for W. Let $\alpha_0 w_0 + \ldots + \alpha_n w_n = 0$. Multiply both sides by $(A - \lambda_1)^n$ to get $\alpha_0 w_0 = 0$, hence $\alpha_0 = 0$. Now proceed by induction to conclude that $\alpha_i = 0$ for all i.

It is obvious that A acts as a full Jordan block with respect to this basis, hence its eigenspace in W for λ_1 is at most 1-dimensional. Let $W' = W + \operatorname{span}\{ABA\, b\}$. Since $ABA\, b$ is an eigenvector of A corresponding to λ_d, A still cannot have two linearly independent eigenvectors in W'. Hence W' is a proper A-invariant subspace of V, which contains b and $ABA\, b$, contradicting Proposition 2.7. □

Proposition 2.9. *The minimal polynomial of A (and B) is the characteristic polynomial. In other words, the Jordan form of A consists of full Jordan blocks.*

Proof: If A is diagonalizable, the statement follows from Lemma 2.8. If not, assume the minimal polynomial properly divides the characteristic polynomial, hence its degree is at most $d-1$.

Then A has an eigenvalue λ and a corresponding generalized eigenvector a such that $(A-\lambda)^2\, a = 0$, and $a' = (A-\lambda)\, a \neq 0$ is an eigenvector of A. Let $b = ABA\, a'$, and $W = \operatorname{span}\left\{A^i\, b \mid i = 0, \ldots, d-2\right\}$. Clearly, W is a proper A-invariant subspace of V. By Proposition 2.7, W cannot contain a'. Hence it cannot contain a either. Let $W' = W + \operatorname{span}\{a'\}$. Then it is easy to see that $a \notin W'$, hence W' is a proper A invariant subspace that contains b and $\delta^{-1} a' = ABA\, b$. This would contradict Proposition 2.7. □

Lemma 2.10. *Let A and B be in ordered triangular form, and $C \in \mathrm{GL}_d(F)$.*

a) If $A = CBC^{-1}$ then C is upper skew-triangular, that is $C_{ij} = 0$ for $i + j > d + 1$.

b) If $B = CAC^{-1}$ then C is lower skew-triangular, that is $C_{ij} = 0$ for $i + j < d + 1$.

In particular AB is lower skew-triangular, and BA is upper skew-triangular.

Proof: Let $\{v_1, \ldots, v_d\}$ be the standard basis for $V = F^d$, and let $V_n = \operatorname{span}\{v_1, \ldots, v_n\}$ and $W_n = \operatorname{span}\{v_{d-n+1}, \ldots, v_d\}$ for $1 \leq n \leq d$. We will prove that $A = CBC^{-1}$ implies $C\, W_n = V_n$ by induction. By the upper-triangular shape of A, v_1 is an eigenvector of A with eigenvalue λ_1 and v_d is an eigenvector of B with the same eigenvalue. Also, $C^{-1}\, v_1$ is an eigenvector of B with eigenvalue λ_1. We can conclude $\operatorname{span}\left\{C^{-1}\, v_1\right\} = \operatorname{span}\{v_d\}$, by Proposition 2.9. This establishes the base case.

Let λ_{n+1} occur k times among $\lambda_1, \ldots, \lambda_n$. By Proposition 2.9, A acts as a full Jordan block on its generalized eigenspaces. A has k generalized eigenvectors with eigenvalue λ_{n+1} in V_n, so they are all in the null space of $(A - \lambda_{n+1})^k$. But A has $k+1$ generalized eigenvectors in V_{n+1}, so $(A - \lambda_{n+1})^k\, v_{n+1} \neq 0$. An analogous argument shows that $(B - \lambda_{n+1})^k\, v_{d-n} \neq 0$, but $(B - \lambda_{n+1})^{k+1} v_{d-n} = 0$. But B also consists of full Jordan blocks, so any vector with this property must be in $\operatorname{span}\{v_{d-n}\}$. Clearly, $C^{-1}\, v_{n+1}$ is such a vector, so $\operatorname{span}\left\{C^{-1}\, v_{n+1}\right\} = \operatorname{span}\{v_{d-n}\}$. We can now use the inductive hypothesis to conclude

$$C^{-1}\, V_{n+1} = C^{-1}\, V_n + C^{-1} \operatorname{span}\{v_{n+1}\} = W_n + \operatorname{span}\{v_{d-n}\} = W_{n+1}.$$

Hence $C\, W_n = V_n$. □

From now on, let us denote $d+1-i$ by $\bar{i}$.

Lemma 2.11. *Let $A, B \in \mathrm{GL}_d(F)$ such that A and B are in ordered triangular form with eigenvalues $\lambda_1, \ldots, \lambda_d \in F^*$. The following are equivalent:*

a) There exists $S \in \mathrm{GL}_d(F)$ skew-diagonal with $S^2 = \gamma I$ for some $\gamma \in F^*$ such that $B = SAS^{-1}$, $(AB)_{ij} = 0$ for $i+j < d+1$ and $\lambda_i \, (AB)_{\bar{i}i} = S_{i\bar{i}}$.

b) There exists $S \in \mathrm{GL}_d(F)$ skew-diagonal with $S^2 = \gamma I$ for some $\gamma \in F^*$ such that $B = SAS^{-1}$, $(BA)_{ij} = 0$ for $i+j > d+1$ and $\lambda_i \, (BA)_{i\bar{i}} = S_{i\bar{i}}$.

c) A and B satisfy the braid relation $ABA = BAB$.

Proof: For $(a) \Leftrightarrow (b)$, note that $BA = S(AB)\, S^{-1}$, hence

$$(BA)_{ij} = S_{i\bar{i}} \, (AB)_{\bar{i}\bar{j}} \, S^{-1}_{\bar{j}j}.$$

For $(b) \implies (c)$, we already know AB is lower skew-triangular by $(b) \implies (a)$. Hence $A(BA)$ is upper skew-triangular, and $(AB)\, A$ is lower skew-triangular, that is ABA is skew-diagonal. Also $(ABA)_{i\bar{i}} = \lambda_i \, (BA)_{i\bar{i}} = S_{i\bar{i}}$, thus $ABA = S$. Hence

$$BAB = S\,(ABA)\, S^{-1} = S = ABA.$$

$(c) \implies (b)$ follows by setting $S = ABA$ and noting that S and BA have the desired properties by Lemma 2.10 and by the upper-triangularity of A. $\square$

2.4 The Matrices

We are now ready to prove that we can always choose a basis of V which makes A and B ordered triangular.

Lemma 2.12. *The set $\left\{a_i \mid 1 \le i \le \left[\frac{d+1}{2}\right]\right\} \cup \left\{b_i \mid 1 \le i \le \left[\frac{d-1}{2}\right]\right\}$ is a basis of V.*

Proof: If $d = 2$, $\operatorname{span}\{a_1\} \neq \operatorname{span}\{b_1\}$, by Lemma 2.5. Hence

$$\dim \operatorname{span}\{a_1, b_1\} \ge 2.$$

If $d = 3$, then $b_1 \notin \operatorname{span}\{a_1, a_2\}$, by Proposition 2.7. Hence

$$\dim\{a_1, a_2, b_1\} > \dim\{a_1, a_2\} = 2.$$

If $d = 4$, let $V_1 = \{a_1, a_2\}$, $V_2 = V_1 + (ABA)\,V_1$, $V_3 = V_1 \cap (ABA)\,V_1$. If $V_1 = V_3$, then V_1 is invariant under $\langle A,\, ABA\rangle$. Hence $\dim V_3 < \dim V_1$ and $\dim V_2 = \dim V_1 + \dim(ABA)\,V_1 - \dim V_3 = 2\dim V_1 - \dim V_3 > \dim V_1 = 2$.

If $\dim V_2 = 3$, then $\dim V_3 = 1$ and V_3 is spanned by some eigenvector w of ABA. If $a_1 \in V_3$ then $V_3 = \operatorname{span}\{a_1\}$, so V_3 would be invariant under $\langle A,\, ABA\rangle$. Hence $V_1 = \operatorname{span}\{a_1, w\}$ and $(ABA)\,V_1 = \operatorname{span}\{b_1, w\}$.

Note that $V_3 \subseteq V_1$, so $(AB)\,V_3 \subseteq (AB)\,V_1 = (ABA)\,V_1$. If $(AB)\,V_3 = V_3$, then V_3 would be $\langle AB,\, ABA\rangle$-invariant. Hence $(ABA)\,V_1 = V_3 + (AB)\,V_3$.

Similarly, $V_3 \subseteq (ABA)\,V_1$, so $(AB)^2\,V_3 = A(BAB)\,V_3 = A\,V_3 \subseteq A\,V_1 = V_1$. If $A\,V_3 = (AB)^2\,V_3 \subseteq V_3$, then V_3 would be $\langle A,\, ABA\rangle$-invariant. Hence $V_1 = V_3 + (AB)^2\,V_3$.

So $V_2 = V_1 + (ABA)\,V_1 = V_3 + (AB)\,V_3 + (AB)^2\,V_3$ and is invariant under $\langle ABA,\, AB\rangle$.

If $d = 5$, we know $\operatorname{span}\{a_1, a_2, b_1, b_2\}$ is 4-dimensional, by the same argument as in the previous case. Let $V_1 = \operatorname{span}\{a_1, a_2, a_3\}$, $V_2 = V_1 + (ABA)\,V_1$ and $V_3 = V_1 \cap (ABA)\,V_1$. If $a_3 \in \operatorname{span}\{a_1, a_2, b_1, b_2\}$, then $b_3 = (ABA)\,a_3 \in \operatorname{span}\{a_1, a_2, b_1, b_2\}$ too. Hence $\dim V_2 = 4$ and $\dim V_3 = \dim V_1 + \dim(ABA)\,V_1 - \dim V_2 = 2$.

Since $V_3 \subsetneq V_1$, there exists $1 \le i \le 3$ such that $a_i \notin V_3$. Thus $V_1 = \operatorname{span}\{a_i\} + V_3$ and $(ABA)\,V_1 = \operatorname{span}\{b_i\} + V_3$.

Like before, $V_3 \subseteq V_1$, so $(AB)\,V_3 \subseteq (AB)\,V_1 = (ABA)\,V_1$ and $(AB)^2\,V_3 = A(ABA)\,V_3 = A\,V_3 \subseteq A\,V_1 = V_1$. If $(AB)\,V_3 = V_3$ then V_3 would be $\langle AB,\, ABA\rangle$-invariant. Hence $(ABA)\,V_1 = V_3 + (AB)\,V_3$. Analogously, if $A\,V_3 = (AB)^2\,V_3 = V_3$, then V_3 would be $\langle A, ABA\rangle$-invariant. Therefore $V_1 = V_3 + (AB)^2\,V_3$.

Again, $V_2 = V_1 + (ABA)\,V_1 = V_3 + (AB)\,V_3 + (AB)^2\,V_3$ and is a proper subspace invariant under $\langle ABA,\, AB\rangle$. □

Proposition 2.13. *If V is a simple B_3-module of dimension $d \le 5$, then there is a basis of V that makes A and B ordered triangular.*

Proof: If $d = 2$, then $\{a_1, b_1\}$ is a basis of V by Lemma 2.12 and it is clear that A and B are ordered triangular with respect to this basis.

If $d = 3$, let $v_1 = a_1$ and $v_3 = b_1$. By Lemma 2.12, $\{a_1, a_2, b_1\}$ is a basis of V. By Lemma 2.7, $a_1 \notin \operatorname{span}\{b_1, b_2\}$, hence $a_2 = \alpha a_1 + \beta_1 b_1 + \beta_2 b_2$ for some α, β_1, β_2. Let $v_2 = a_2 - \alpha a_1$. Then $\operatorname{span}\{a_1, v_2\} = \operatorname{span}\{a_1, a_2\}$, so $\{a_1, v_2, b_1\}$ is still a basis of V. But $v_2 \in \operatorname{span}\{a_1, a_2\} \cap \operatorname{span}\{b_1, b_2\}$ by construction, so $A v_2 \in \operatorname{span}\{a_1, a_2\}$ and $Bv_2 \in \operatorname{span}\{b_1, b_2\}$, which shows that A and B are ordered triangular with respect to $\{v_1, v_2, v_3\}$.

If $d = 4$, let $v_1 = a_1$ and $v_4 = b_1$. We can construct v_2 similarly as in the previous case by noting that $a_1 \notin \operatorname{span}\{b_1, b_2, b_3\}$ by Lemma 2.7, so there exists α such that $a_2 - \alpha a_1 \in \operatorname{span}\{b_1, b_2, b_3\}$. Let $v_3 = AB\, v_2$.

Note that $\operatorname{span}\{a_1, v_2\} = \operatorname{span}\{a_1, a_2\}$ and by letting ABA act on both sides we also have $\operatorname{span}\{b_1, v_3\} = \operatorname{span}\{b_1, b_2\}$. Hence $V = \operatorname{span}\{v_1, v_2, v_3, v_4\}$. Since $v_2 \in \operatorname{span}\{a_1, a_2\} \cap \operatorname{span}\{b_1, b_2, b_3\}$ we also have $v_3 = ABA\, v_2 \in \operatorname{span}\{b_1, b_2\} \cap \operatorname{span}\{a_1, a_2, a_3\}$. This shows that A and B are ordered triangular with respect to $\{v_1, v_2, v_3, v_4\}$.

If $d = 5$, let $v_1 = a_1$ and $v_5 = b_1$. Now follow the method in the previous case to construct $v_2 = a_2 - \alpha a_1 \in \operatorname{span}\{a_1, a_2\} \cap \operatorname{span}\{b_1, b_2, b_3, b_4\}$. Let $v_4 = ABA\, v_2 \in v_2 \in \operatorname{span}\{b_1, b_2\} \cap \operatorname{span}\{a_1, a_2, a_3, a_4\}$. By Lemma 2.12, $a_3 = \alpha_1 a_1 + \alpha_2 a_2 + \beta_1 b_1 + \beta_2 b_2$. Let $v_3 = a_3 - \alpha_1 a_1 - \alpha_2 a_2$. Then $v_3 \in \operatorname{span}\{a_1, a_2, a_3\} \cap \operatorname{span}\{b_1, b_2, b_3\}$.

Note that $\operatorname{span}\{v_1, v_2, v_3\} = \operatorname{span}\{a_1, a_2, a_3\}$ and $\operatorname{span}\{v_1, v_2\} = \operatorname{span}\{a_1, a_2\}$ by construction of v_2 and v_3. Acting on both sides of the second equality by ABA yields $\operatorname{span}\{v_5, v_4\} = \operatorname{span}\{b_1, b_2\}$. Hence $\operatorname{span}\{v_1, \ldots, v_5\} = \operatorname{span}\{a_1, a_2, a_3, b_1, b_2\} = V$. So $\{v_1, \ldots, v_5\}$ is a basis of V that makes A and B ordered triangular. $\square$

Actually, we can make A and B look even more special and the computation simpler without losing generality.

Corollary 2.14. *If V is a simple B_3-module of dimension $d \leq 5$, then there is a basis of V that makes A and B ordered triangular and $B = SAS^{-1}$ for S skew-diagonal and $S_{i\bar{i}} = 1$.*

Proof: Choose a basis $\{v_1, \ldots, v_d\}$ such that A and B are ordered triangular. As in the proof of Lemma 2.11, we know that ABA is skew-diagonal and $(ABA)^2 = \delta$. If d is odd, let $\gamma = (ABA)_{[\frac{d+1}{2}],[\frac{d+1}{2}]}$ and note that $\gamma^2 = \delta$, otherwise pick γ to be any square root of δ. Let $S = \gamma^{-1}ABA$. Then $S^2 = I$, so $S_{i\bar{i}}S_{\bar{i}i} = 1$. For $i = 1, \ldots, \left[\frac{d}{2}\right]$, let $v_i \mapsto S_{i\bar{i}}v_i$. Observe that for this new basis, $Sv_i = v_{\bar{i}}$. □

From now on, let A, B, and S be as in Corollary 2.14.

Lemma 2.15. *If BA is upper skew-triangular, A and B satisfy the braid relation.*

Proof: Note $S^{-1} = S$, so $B = SAS$. Just like in the proof of Lemma 2.11, if $BA = (SAS)A$ is upper skew-triangular, $AB = A(SAS)$ is lower skew-triangular and $ABA = A(SAS)A$ is skew-diagonal. So is $BAB = S(ABA)S$. Hence

$$(ABA)_{i\bar{i}} = A_{ii}(SASA)_{i\bar{i}} = \lambda_i(SASA)_{i\bar{i}}$$

and

$$(BAB)_{i\bar{i}} = (SASA\,B)_{i\bar{i}} = (SASA)_{i\bar{i}}B_{\bar{i}\bar{i}} = (SASA)_{i\bar{i}}\lambda_i.$$

Hence $ABA = BAB$. □

Lemma 2.16. *Let V be a simple B_3-module of dimension $d \leq 5$ and $\{v_1, \ldots, v_d\}$ a basis as in Corollary 2.14. Let D be a diagonal matrix such that $D_{i,i} = D_{\bar{i}\bar{i}}$ for all i, and $A' = DAD^{-1}$ and $B' = DBD^{-1}$. Then A' and B' are still ordered triangular and $B' = SA'S^{-1}$.*

Proof: Note that D corresponds only to a scaling of the basis vectors, so conjugating by D does not change the triangular shapes and the diagonal entries of A and B. By direct computation, $DS = DS$, hence

$$B' = DBD^{-1} = DSAS^{-1}D^{-1} = SDAD^{-1}S^{-1} = SA'S^{-1}.$$

□

Now we are ready to start computing the representations. By Lemma 2.11, $(BA)_{ij} = 0$ for $i + j > d$ is a necessary condition for $ABA = BAB$ and by Lemma 2.15 it is sufficient too.

Proposition 2.17. *Let V be a simple 2-dimensional B_3-module. Then there exists a basis of V for which*

$$A = \begin{pmatrix} \lambda_1 & \lambda_1 \\ 0 & \lambda_2 \end{pmatrix} \quad B = \begin{pmatrix} \lambda_2 & 0 \\ -\lambda_2 & \lambda_1 \end{pmatrix}.$$

Proof: We set $0 = (BA)_{22} = A_{12}^2 + \lambda_1\lambda_2$, hence $A_{12} = \sqrt{-\lambda_1\lambda_2}$.

Now rescale $v_1 \mapsto \frac{A_{12}}{\lambda_1} v_1$ to obtain A and B in the above form. $\square$

Proposition 2.18. *Let V be a simple 3-dimensional B_3-module. Then there exists a basis of V for which*

$$A = \begin{pmatrix} \lambda_1 & -\lambda_1 - \frac{\lambda_2^2}{\lambda_3} & -\lambda_2 \\ 0 & \lambda_2 & \lambda_3 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad B = \begin{pmatrix} \lambda_3 & 0 & 0 \\ \lambda_3 & \lambda_2 & 0 \\ -\lambda_2 & -\lambda_1 - \frac{\lambda_2^2}{\lambda_3} & \lambda_1 \end{pmatrix}.$$

Proof: Note that if $A_{23} = 0$, then $B_{21} = 0$ and span$\{v_1, v_3\}$ would be invariant. So $A_{23} \neq 0$, and we can let

$$D = \begin{pmatrix} 1 & & \\ & \frac{\lambda_3}{A_{23}} & \\ & & 1 \end{pmatrix}$$

and we can replace A by DAD^{-1}, and B by DBD^{-1} by Lemma 2.16.

Now

$$A = \begin{pmatrix} \lambda_1 & A_{12} & A_{13} \\ & \lambda_2 & \lambda_3 \\ & & \lambda_3 \end{pmatrix} \qquad B = \begin{pmatrix} \lambda_3 & & \\ \lambda_3 & \lambda_2 & \\ A_{13} & A_{12} & \lambda_1 \end{pmatrix}.$$

Hence

$$0 = (BA)_{23} = \lambda_3 A_{13} + \lambda_2\lambda_3$$

forces $A_{13} = -\lambda_2$, and

$$0 = (BA)_{33} = A_{13}^2 + A_{12}\,A_{23} + \lambda_1\lambda_3 = \lambda_2^2 + \lambda_3\,A_{12} + \lambda_1\lambda_3$$

forces $A_{12} = -\lambda_1 - \lambda_2^2\lambda_3^{-1}$. $\square$

Proposition 2.19. *Let V be a simple 4-dimensional B_3-module and $D = -\sqrt{\lambda_1\lambda_4/\lambda_2\lambda_3}$. Then there exists a basis of V for which*

$$A = \begin{pmatrix} \lambda_1 & \lambda_2(1+D+D^2) & \lambda_3(1+D+D^2) & \lambda_4 \\ 0 & \lambda_2 & \lambda_3(1+D) & \lambda_4 \\ 0 & 0 & \lambda_3 & \lambda_4 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$$

$$B = \begin{pmatrix} \lambda_4 & 0 & 0 & 0 \\ -\lambda_3 & \lambda_3 & 0 & 0 \\ \lambda_2 D^{-1} & -\lambda_2(1+D^{-1}) & \lambda_2 & 0 \\ -\lambda_1 D^{-3} & \lambda_1(D^{-1}+D^{-2}+D^{-3}) & -\lambda_1(1+D^{-1}+D^{-2}) & \lambda_1 \end{pmatrix}.$$

Proof: If $A_{24} = 0$, then $0 = (BA)_{34} = A_{14}A_{24} + A_{23}A_{24} + \lambda_2 A_{34}$ forces $A_{34} = 0$. But span $\{v_1, v_4\}$ would then be invariant. By Lemma 2.16, we can conjugate A and B by

$$D = \begin{pmatrix} 1 & & & \\ & \lambda_4 A_{24}^{-1} & & \\ & & \lambda_4 A_{24}^{-1} & \\ & & & 1 \end{pmatrix}$$

to get

$$A = \begin{pmatrix} \lambda_1 & A_{12} & A_{13} & A_{14} \\ & \lambda_2 & A_{23} & \lambda_4 \\ & & \lambda_3 & A_{34} \\ & & & \lambda_4 \end{pmatrix} \qquad B = \begin{pmatrix} \lambda_4 & & & \\ A_{34} & \lambda_3 & & \\ \lambda_4 & A_{23} & \lambda_2 & \\ A_{14} & A_{13} & A_{12} & \lambda_1 \end{pmatrix}.$$

It follows from

$$0 = (BA)_{24} = A_{14}A_{34} + \lambda_3\lambda_4$$

that $A_{14} \neq 0$ and $A_{34} = -\lambda_3\lambda_4 {A_{14}}^{-1}$.

Now

$$0 = (BA)_{33} = \lambda_4 A_{13} + {A_{23}}^2 + \lambda_2\lambda_3 \implies A_{13} = -\frac{A_{23}^2 + \lambda_2\lambda_3}{\lambda_4}$$

and

$$0 = (BA)_{34} = \lambda_4\left(A_{14} + A_{23} - \frac{\lambda_2\lambda_3}{A_{14}}\right) \implies A_{23} = -A_{14} + \frac{\lambda_2\lambda_3}{A_{14}}.$$

Also

$$0 = (BA)_{42} = A_{14}A_{12} - \lambda_2{\lambda_4^{-1}}A_{14}^2 + \lambda_2^2\lambda_3\lambda_4^{-1} - \frac{\lambda_2^3\lambda_3^2}{\lambda_4 A_{14}^2} = 0 \implies$$

$$A_{12} = \lambda_2{\lambda_4^{-1}}A_{14} - \frac{\lambda_2^2\lambda_3}{\lambda_4 A_{14}} + \frac{\lambda_2^3\lambda_3^2}{\lambda_4 A_{14}^3}.$$

And finally

$$0 = (BA)_{44} = \lambda_1\lambda_4 - \frac{\lambda_2^3\lambda_3^3}{A_{14}^4} \implies A_{14} = \sqrt[4]{\frac{\lambda_2^3\lambda_3^3}{\lambda_1\lambda_4}}.$$

Now rescale $v_i \mapsto A_{i4}/\lambda_4 v_i$ and substitute $D = -\sqrt{\lambda_1\lambda_4/\lambda_2\lambda_3}$ to obtain A and B in the desired form. $\square$

The computation proceeds similarly for $d = 5$, only the matrix entries turn out to be more complicated. Therefore we will omit listing the actual matrices.

Proposition 2.20. *Let V be a simple 5-dimensional B_3-module and $D = \sqrt[5]{\det A}$. Then for each choice of D, there exists a basis of V for which A and B are in ordered triangular form and the B_3 action is unique up to conjugation.*

Proof: Note that $A_{15} \neq 0$, otherwise $(BA)_{15} = \lambda_5 A_{15} = 0$, which would eventually make $(ABA)_{15} = 0$ and ABA singular.

If $A_{35} = 0$, then

$$0 = (BA)_{35} = A_{35}A_{15} + A_{34}A_{25} + \lambda_3 A_{35}$$

implies either $A_{34} = 0$ or $A_{25} = 0$. If $A_{34} = 0$, then $B_{31} = B_{32} = 0$ too, so span $\{v_1, v_2, v_4, v_5\}$ would be B_3-invariant. Hence $A_{25} = 0$ and

$$0 = (BA)_{25} = A_{45}A_{15} + \lambda_4 A_{25}$$

and $A_{45} = 0$ as $A_{15} \neq 0$. If $A_{45} = 0$, then $B_{21} = B_{31} = B_{41} = 0$, so span $\{v_1, v_5\}$ would be B_3-invariant. Hence $A_{35} \neq 0$.

If $A_{45} = 0$, then $(BA)_{25} = 0$ would make $A_{25} = 0$ too and

$$0 = (BA)_{45} = A_{25}A_{15} + A_{24}A_{25} + A_{23}A_{35} + \lambda_2 A_{45}$$

would imply $A_{23} = 0$. Hence $B_{21} = B_{41} = B_{43} = 0$, and span $\{v_1, v_3, v_5\}$ would be B_3-invariant. So $A_{45} \neq 0$.

Hence we can rescale

$$\begin{aligned} v_4 &\mapsto -\frac{A_{45}}{\lambda_4} v_4 \\ v_3 &\mapsto \frac{A_{35}}{A_{15}} v_3 \end{aligned}$$

With respect to this new basis, $A_{45} = -\lambda_4$ and $A_{35} = A_{15}$.

From $(BA)_{25} = 0$, $A_{25} = -A_{15}$ and now

$$0 = (BA)_{35} = A_{15}(A_{15} - A_{34} + \lambda_3)$$

implies $A_{34} = A_{15} + \lambda_3$.

Setting

$$0 = (BA)_{52} = A_{15}A_{12} + \lambda_2 A_{14}$$

shows that if $A_{12} = 0$ then $A_{14} = 0$ and

$$0 = (BA)_{53} = A_{13}(A_{15} + \lambda_3).$$

If $A_{13} = 0$, then $B_{52} = B_{53} = B_{54} = 0$ and hence $\operatorname{span}\{v_2, v_3, v_4\}$ is B_3-invariant. If $A_{15} = -\lambda_3$, then $A_{34} = 0$ and in turn $B_{32} = B_{54} = B_{52} = 0$ making $\operatorname{span}\{v_2, v_4\}$ a proper B_3-invariant subspace. Hence we can assume $A_{12} \neq 0$ and

$$A_{14} = -\frac{A_{15}A_{12}}{\lambda_2}.$$

Now

$$0 = (BA)_{34} = -\frac{A_{15}^2 A_{12}}{\lambda_2} + A_{24}A_{15} + \lambda_3 A_{24} + \lambda_3 A_{15} + \lambda_3^2$$

forces

$$A_{12} = \frac{\lambda_2}{A_{15}^2}(A_{24} + \lambda_3)(A_{15} + \lambda_3)$$

and

$$(BA)_{53} = \frac{(A_{15} + \lambda_3)(A_{15}A_{13} + A_{24}A_{23} + A_{23}\lambda_3)}{A_{15}}.$$

If $A_{15} = -\lambda_3$, then $A_{12} = 0$ by $(BA)_{34} = 0$, which we have ruled out already. Therefore

$$A_{13} = \frac{(A_{24} + \lambda_3)A_{23}}{A_{15}}.$$

Letting

$$0 = (BA)_{45} = -A_{15}^2 - A_{24}A_{15} + A_{23}A_{15} + \lambda_4\lambda_2$$

implies

$$A_{23} = A_{15} + A_{24} - \frac{\lambda_2\lambda_4}{A_{15}}.$$

To solve for A_{24}, set

$$0 = (BA)_{55} - (BA)_{44} = \frac{\lambda_2\lambda_3\lambda_4 A_{24} + \lambda_2\lambda_3^2\lambda_4 + \lambda_1\lambda_5 A_{15}^2 + \lambda_2\lambda_3\lambda_4 A_{15}}{A_{15}^2}$$

to obtain

$$A_{24} = -A_{15} - \lambda_3 - \frac{\lambda_1\lambda_5 A_{15}^2}{\lambda_2\lambda_3\lambda_4}.$$

Finally substitute this back in

$$0 = (BA)_{44} = \frac{\lambda_1^2 \lambda_5^2 A_{15}^5 - \lambda_2^3 \lambda_3^3 \lambda_4^3}{\lambda_2^2 \lambda_3^2 \lambda_4^2 A_{15}}$$

to get

$$A_{15} = \frac{\sqrt[5]{\lambda_1^3 \lambda_2^3 \lambda_3^3 \lambda_4^3 \lambda_5^3}}{\lambda_1 \lambda_5}.$$

$\square$

2.5 Characterization of All Simple Representations

Now that we know that simple representations of dimension $d \leq 5$ of B_3 are of the form described in the last four theorems, the natural question to ask is whether all representations of this form are simple. We will find that the answer is no, but we can give an explicit necessary and sufficient condition for simplicity in terms of the eigenvalues.

Let $P_n^{(d)}(x) = \prod_{i \neq n}(x - \lambda_i)$ for $1 \leq n \leq d$. Note that $(x - \lambda_n)\, P_n^{(d)}(x)$ is the characteristic polynomial of A and B. Hence $(A - \lambda_n)\, P_n^{(d)}(A) = 0$ so the columns of $P_n^{(d)}(A)$ are 0 or eigenvectors of A. By Proposition 2.9, $P_n^{(d)}(A) \neq 0$ so at least one of the columns is nonzero and the others are multiples of this column. So $P_n^{(d)}(A)$ is of rank 1. Analogous statements hold for $P_n^{(d)}(B)$. Hence

$$P_n^{(d)}(A)\, P_m^{(d)}(B)\, P_n^{(d)}(A) = Q_{mn}^{(d)}\, P_n^{(d)}(A)$$

for some constant $Q_{mn}^{(d)}$. A and B can be switched in the last equation by conjugating by ABA. The entries of A and B are rational functions in $\lambda_1, \ldots, \lambda_d, \delta$, therefore the $Q_{mn}^{(d)}$ are also rational functions of the same variables.

Denote by $E_{ij}^{(d)}$ the elementary $d \times d$ matrix whose only nonzero entry is a 1 in the (i, j) position.

Lemma 2.21.

a) $P_1^{(d)}(B)\, P_d^{(d)}(A) = Q_{1d}^{(d)}\, E_{dd}^{(d)}$.

b) $P_m^{(d)}(B)\, P_n^{(d)}(A) = 0$ *if and only if* $Q_{mn}^{(d)} = 0$.

c) The polynomials are

$$Q^{(2)}_{mn} = -\lambda_m^2 + \lambda_m \lambda_n - \lambda_n^2$$

$$Q^{(3)}_{mn} = (\lambda_m^2 + \lambda_n \lambda_k)(\lambda_n^2 + \lambda_m \lambda_k)$$

with $k \neq m, n$.

$$Q^{(4)}_{mn} = -\gamma^{-1}(\lambda_m^2 + \gamma)(\lambda_n^2 + \gamma)(\gamma + \lambda_m \lambda_k + \lambda_n \lambda_l)(\gamma + \lambda_m \lambda_l + \lambda_n \lambda_k)$$

with $\gamma^2 = \lambda_1 \cdots \lambda_4$ and $k, l \neq m, n$.

$$Q^{(5)}_{mn} = \gamma^{-8}(\gamma^2 + \lambda_m \gamma + \lambda_m^2)(\gamma^2 + \lambda_n \gamma + \lambda_n^2) \prod_{k \neq m,n} (\gamma^2 + \lambda_m \lambda_k)(\gamma^2 + \lambda_n \lambda_k)$$

with $\gamma^5 = \lambda_1 \cdots \lambda_5$.

Proof:

a) Observe that $(B - \lambda_i)\, v_j \in \operatorname{span}\{v_{i+1}, \ldots, v_d\}$ for all $j \geq i$. Hence $P^{(d)}_1(B)V \subseteq \operatorname{span}\{v_d\}$, that is the only nonzero entries of $P^{(d)}_1(B)$ are in the bottom row.

Also, $(A - \lambda_i)\, v_j \in \operatorname{span}\{v_1, \ldots, v_{i-1}\}$ for all $j \leq i$. Hence $P^{(d)}_d(A)\, v_j = 0$ for $j < d$, and $P^{(d)}_d\, (A) v_d \in \operatorname{span}\{v_1\}$, that is the only nonzero entry of $P^{(d)}_d(A)$ is in the top right corner. So $P^{(d)}_d(A) = \alpha\, E^{(d)}_{1d}$ and $P^{(d)}_1(B)\, P^{(d)}_d(A) = \beta\, E^{(d)}_{dd}$ for some $\alpha, \beta \in F$. So

$$\begin{aligned} P^{(d)}_d(A)\, P^{(d)}_1(B)\, P^{(d)}_d(A) &= \alpha\beta\, E^{(d)}_{1d} \\ Q^{(d)}_{1d}\, P^{(d)}_d(A) = Q^{(d)}_{1d}\, \beta\, E^{(d)}_{1d} & \end{aligned}$$

and hence $\alpha = Q^{(d)}_{1d}$.

b) This follows from a) by reindexing the eigenvalues.

c) Using a), $Q^{(d)}_{1d}$ can be easily found by direct computation. Then just reindex the eigenvalues so that 1 and d are replaced by m and n for the general case.

$\square$

Theorem 2.22. *Let F be an algebraically closed field.*

a) There exists a simple representation of B_3 *on a vector space* V *of* F*-dimension* $d \leq 5$ *if and only if the eigenvalues and (for* $d = 4, 5$*)* γ*, as defined in Lemma 2.21, satisfy* $Q_{mn}^{(d)} \neq 0$ *for* $1 \leq m, n \leq d$*.*

b) Any simple B_3*-module over* F *is uniquely determined by the eigenvalues and (for* $d = 4, 5$*) by a choice of the root* γ*.*

Proof:

a) Assume V is a simple B_3-module. Then, as we have already observed, $P_n^{(d)}(A)$ is nonzero for all $1 \leq n \leq d$. Suppose $Q_{mn}^{(d)} = 0$ for some m, n. Then $P_m^{(d)}(B)\, P_n^{(d)}(A) = 0$ by Lemma 2.21. Since $P_n^{(d)}(A) \neq 0$, there exists a vector $v \in V$ such that $a_n = P_n^{(d)}(A)\, v \neq 0$, hence a_n is an eigenvector of A with eigenvalue λ_n. Let $b_n = ABA\, a_n$, which is then an eigenvector of B with the same eigenvalue. Since $P_m^{(d)}(B)\, a_n = P_m^{(d)}(B)\, P_n^{(d)}(A)\, v = 0$, $W = \text{span}\left\{a_n, B\, a_n, \ldots, B^{d-2}\, a_n, b_n\right\}$ is B-invariant. Observe that

$$P_m^{(d)}(B)\, B^i\, a_n = B_i\, P_m^{(d)}(B)\, a_n = 0$$

and

$$P_m^{(d)}(B)\, b_n = \prod_{i \neq m,\, n} (B - \lambda_i)\ (B - \lambda_n) b_n = 0.$$

Thus $P_m^{(d)}(B)$ restricted to W is 0, which shows W is a proper subspace of V. This contradicts Proposition 2.7.

Conversely, let $Q_{mn}^{(d)} \neq 0$ for all $m \neq n$. Let $W \subseteq V$ be a nonzero B_3-submodule. Then A has an eigenvector a_i in W. We know by Lemma 2.21 that $P_n^{(d)}(A) \neq 0$ for $1 \leq n \leq d$, so the minimal polynomial of A is the characteristic polynomial, hence the Jordan form of A contains only full blocks, so its eigenspaces are 1-dimensional. Since $P_i^{(d)}(A) \neq 0$, there exists $v \in V$ such that $P_i^{(d)}(A)\, v \neq 0$. So $P_i^{(d)}(A)\, v$ is an eigenvector of A with eigenvalue λ_i, just like a_i. Hence $a_i \in \text{span}\left\{P_i^{(d)}(A)\, v\right\}$ and we can always scale v so that $a_i = P_i^{(d)}(A)\, v$.

We will now show that v_1, $v_d \in W$. If $i = 1$, then

$$v_d = ABA\, v_1 \in (ABA)\, W = W.$$

If not, let $b_1 = P_1^{(d)}(B)\, P_i^{(d)}(A)\, v$. Now

$$P_i^{(d)}(A)\, b_1 = P_i^{(d)}(A)\, P_1^{(d)}(B)\, P_i^{(d)}(A)\, v = Q_{1i}^{(d)}\, P_i^{(d)}(A)\, v \neq 0$$

so $b_1 \neq 0$ and b_1 is an eigenvector of B with eigenvalue λ_1. So is v_d, thus $v_d \in \operatorname{span}\{b_1\} \subseteq W$ and we may scale v_d so that $v_d = b_1$. Also, $v_1 = (ABA)^{-1}\, v_d \in W$. Let $w_1 = v_1$, $w_d = v_d$ and $w_i = \prod_{j=i+1}^{d}(A - \lambda_j)\, v_d$ for $2 \leq i < d$. Obviously, $w_i \in W$. Note that

$$P_1^{(d)}(B)\,(A - \lambda_1)(A - \lambda_3) \cdots (A - \lambda_i) w_i = P_1^{(d)}(B)\, P_2^{(d)}(A)\, v_d =$$

$$P_1^{(d)}(B)\, P_2^{(d)}(A) v_d\, P_1^{(d)}(B)\, a_i = Q_{12}^{(d)}\, P_1^{(d)}(B)\, a_i \neq 0$$

so $w_i \neq 0$ for all i. We will prove that they are linearly independent. Let $\sum \alpha_i w_i = 0$ with at least one nonzero coefficient. Let k be maximal with respect to $\alpha_k \neq 0$. If $k = 1$, then $\alpha_1 v_1 = 0$ implies $\alpha_1 = 0$, a contradiction. So $k \geq 2$, and

$$\prod_{j=1}^{k-1}(A - \lambda_j) \sum_{i=1}^{d} \alpha_i w_i = \alpha_k P_k^{(d)}(A)\, v_d = 0.$$

But $P_k^{(d)}(A)\, v_d = P_k^{(d)}(A)\, P_1^{(d)}(B)\, a_i \neq 0$ by the usual argument, so $\alpha_k = 0$, which is a contradiction.

So $\operatorname{span}\{w_1, \ldots, w_d\}$ is a d-dimensional subspace of W. Hence $W = V$.

b) This follows from our earlier computations.

$\square$

Chapter 3

Unitary Representations

3.1 Introduction

Unitary braid representations have been constructed in several ways using the representation theory of Kac-Moody algebras and quantum groups, see e.g. [6], [9], and [16], and specializations of the reduced Burau and Gassner representations in [1]. Such representations easily lead to representations of $\mathrm{PSL}(2, \mathbb{Z}) = B_3/Z$, where Z is the center of B_3, and $\mathrm{PSL}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})/\{\pm 1\}$, where $\{\pm 1\}$ is the center of $\mathrm{SL}(2, \mathbb{Z})$. We give a complete classification of simple unitary representations of B_3 of dimension $d \leq 5$ in this paper. In particular, the unitarizability of a braid representation depends only on the the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_d$ of the images the two generating twists of B_3. The condition for unitarizability is a set of linear inequalities in the logarithms of these eigenvalues. In other words, the representation is unitarizable if and only if the $(\arg \lambda_1, \arg \lambda_2, \ldots, \arg \lambda_d)$ is a point inside a polyhedron in $(\mathbb{R}/2\pi)^d$, where we give the equations of the hyperplanes that bound this polyhedron. This classification shows that the approaches mentioned previously do not produce all possible unitary braid representations. We obtain representations that seem to be new for $d \geq 3$. As any unitary representation of B_n restricts to a unitary representation of B_3 in an obvious way, these results may also be useful in classifying such representation of B_n.

Since we are interested in unitarizable representations, we will let $F = \mathbb{C}$ and we will require that $|\lambda_i| = 1$. Let $\rho : B_3 \to V$ be a simple d-dimensional representation $(d \leq 5)$, and $A = \rho(\sigma_1)$, $B = \rho(\sigma_2)$. Any unitarizable complex matrix is diagonalizable, so we can assume that A and B are diagonalizable. So the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_d$ are distinct by Proposition 2.8. Denote the $\mathbb{C}$-algebra generated by A and B by $\mathcal{B}$. In other words, $\mathcal{B} = \rho(\mathbb{C}\, B_3)$, where $\mathbb{C}\, B_3$ is the group algebra. Note that $\mathcal{B} = \mathrm{End}(V)$ by simplicity.

The proof proceeds by defining a vector space antihomomorphism $\imath : \mathcal{B} \to \mathcal{B}$ and proving that it is an algebra antihomomorphism and an involution of $\mathcal{B}$ in section 3.2. In section 3.3, we define a sesquilinear form $\langle . , . \rangle$ on the ideal $I = \mathcal{B}e_{B,1}$ that is invariant under multiplication by A and B. We prove that $\langle . , . \rangle$ is positive definite if $\mu_{i1}^{(d)} > 0$ for $2 \le i \le d$. In this case, ρ is a unitary representation of B_3 on the d-dimensional vector space I. We also prove that if ρ is a unitarizable representation $\mu_{i1}^{(d)} > 0$ for $2 \le i \le d$. In section 3.4, we give some examples of using the positivity of $\mu_{i1}^{(d)}$.

3.2 An Involution of the Image of B_3

Let $e_{M,i}$ be the eigenprojection of M to the eigenspace of λ_i, where $M \in \{A, B\}$. That is

$$e_{M,i} = \prod_{j \ne i} \frac{M - \lambda_j}{\lambda_i - \lambda_j} = \frac{P_i^{(d)}(M)}{P_i^{(d)}(\lambda_i)}.$$

Note that $e_{A,i}$ and $e_{B,i}$ always exist because the eigenvalues are distinct. Also $e_{M,i}e_{M,j} = \delta_{ij}e_{M,i}$. Define $\mu_{ji}^{(d)}$ by $e_{B,i}e_{A,j}e_{B,i} = \mu_{ji}^{(d)} e_{B,i}$. Note that

$$\mu_{ji}^{(d)} = \frac{Q_{ji}^{(d)}}{P_i^{(d)}(\lambda_i)\, P_j^{(d)}(\lambda_j)}.$$

Lemma 3.1. *The $\mu_{ij}^{(d)}$ are real numbers.*

Proof: For $i \ne j$, the proof is by direct computation using $\overline{\lambda_i} = \lambda_i^{-1}$ and $\overline{\gamma} = \gamma^{-1}$. For example, for $d = 5$:

$$\begin{aligned}
\mu_{ij}^{(d)} &= \frac{(\gamma^2 + \lambda_i\gamma + \lambda_i^2)(\gamma^2 + \lambda_j\gamma + \lambda_j^2)\prod_{k\ne i,j}(\gamma^2 + \lambda_i\lambda_k)(\gamma^2 + \lambda_j\lambda_k)}{\gamma^8 \prod_{k\ne i}(\lambda_i - \lambda_k)\prod_{k\ne j}(\lambda_j - \lambda_k)} \\
&= \frac{(\gamma\lambda_i^{-1} + 1 + \gamma^{-1}\lambda_i)(\gamma\lambda_j^{-1} + 1 + \gamma^{-1}\lambda_j)}{(1 - \lambda_j\lambda_i^{-1})(1 - \lambda_i\lambda_j^{-1})}\ \frac{\prod_{k\ne i,j}(\gamma^2 + \lambda_i\lambda_k)(\gamma^2 + \lambda_j\lambda_k)}{\gamma^6 \prod_{k\ne i,j}(\lambda_i - \lambda_k)(\lambda_j - \lambda_k)}
\end{aligned}$$

The first of the two quotients is easily seen to be real. For the second quotient,

$$\overline{\left(\frac{\prod_{k\neq i,j}(\gamma^2+\lambda_i\lambda_k)(\gamma^2+\lambda_j\lambda_k)}{\gamma^6\prod_{k\neq i,j}(\lambda_i-\lambda_k)(\lambda_j-\lambda_k)}\right)} \quad = \quad \frac{\prod_{k\neq i,j}(\gamma^{-2}+\lambda_i^{-1}\lambda_k^{-1})(\gamma^{-2}+\lambda_j^{-1}\lambda_k^{-1})}{\gamma^{-6}\prod_{k\neq i,j}(\lambda_i^{-1}-\lambda_k^{-1})(\lambda_j^{-1}-\lambda_k^{-1})}$$

Multiply the numerator and the denominator by $\gamma^{12}\lambda_i^3\lambda_j^3\prod_{k\neq i,j}\lambda_k^2$ to see that this is still

$$\frac{\prod_{k\neq i,j}(\gamma^2+\lambda_i\lambda_k)(\gamma^2+\lambda_j\lambda_k)}{\gamma^6\prod_{k\neq i,j}(\lambda_i-\lambda_k)(\lambda_j-\lambda_k)}$$

For the case $i=j$, note that $\sum_{k=1}^{d} e_{A,k}=I$, so

$$\begin{aligned} e_{B,i} &= e_{B,i}Ie_{B,i} \\ &= e_{B,i}\sum_{k=1}^{d} e_{A,k}e_{B,i} \\ &= \sum_{k=1}^{d} e_{B,i}e_{A,k}e_{B,i} \\ &= \sum_{k=1}^{d} \mu_{ki}^{(d)}e_{B,i} \end{aligned}$$

Hence $\sum_{k=1}^{d}\mu_{ki}^{(d)}=1$, and $\mu_{ii}^{(d)}=1-\sum_{k\neq i}\mu_{ki}^{(d)}$ is real. $\square$

Proposition 3.2. *$S=\{e_{A,i}e_{B,1}e_{A,j} \mid 1\leq i,j\leq d,\ i\neq j\}\cup\{e_{A,i}\mid 1\leq i\leq d\}$ is a basis for the $\mathbb{C}$-vector space $\mathcal{B}$.*

Proof: Suppose

$$\sum_{i=1}^{d}\sum_{\substack{j=1\\ j\neq i}}^{d}\alpha_{ij}e_{A,i}e_{B,1}e_{A,j}+\sum_{i=1}^{d}\alpha_{ii}e_{A,i}=0$$

Multiply by $e_{A,i}$ both on the left and on the right. The only term of the sum that survives is

$$\alpha_{ii}e_{A,i}=0$$

Let v_i be an eigenvector of A corresponding to λ_i. Then $e_{A,i}v_i=v_i\neq 0$, so $e_{A,i}\neq 0$. Hence $\alpha_{ii}=0$.

For $i\neq j$, multiplying by $e_{A,i}$ on the left and by $e_{A,j}$ on the right shows

$$\alpha_{ij}e_{A,i}e_{B,1}e_{A,j}=0$$

But

$$e_{B,1}e_{A,i}e_{B,1}e_{A,j}e_{B,1} = (e_{B,1}e_{A,i}e_{B,1})(e_{B,1}e_{A,j}e_{B,1}) = \mu_{j1}^{(d)}\mu_{i1}^{(d)}e_{B,1} \neq 0$$

so $e_{A,i}e_{B,1}e_{A,j} \neq 0$. Hence $\alpha_{ij} = 0$. So S is linearly independent. It has d^2 elements, hence it is a basis of the d^2-dimensional space $\mathcal{B}$. □

Note: if we know $\mu_{ii}^{(d)} \neq 0$ for all i, we can use the basis $S' = \{e_{A,i}e_{B,1}e_{A,j} \mid 1 \leq i, j \leq d\}$ instead of S. As $e_{A,i}e_{B,1}e_{A,i} = \mu_{ii}^{(d)}e_{A,i}$, S' is almost the same as S. Since S' is more symmetric than S, its use makes the following computations simpler and the arguments more transparent. In the most general case however, $\mu_{ii}^{(d)}$ could be 0.

Define $\imath : \mathbb{C} \to \mathbb{C}$ as the usual complex conjugation. Extend $\imath$ to $\mathcal{B} \to \mathcal{B}$ by requiring $\imath$ to be an antilinear map with $\imath(e_{A,i}) = e_{A,i}$ and $\imath(e_{A,i}e_{B,1}e_{A,j}) = e_{A,j}e_{B,1}e_{A,i}$ for $i \neq j$. Note that $\imath(\mu_{ij}^{(d)}) = \mu_{ij}^{(d)}$.

Lemma 3.3. *$\imath$ as defined above is an antihomomorphism on the algebra $\mathcal{B}$ and $\imath^2 = \mathrm{Id}_{\mathcal{B}}$.*

Proof: It is sufficient to prove that $\imath$ acts as an antihomomorphism on the elements of the basis S. S has two different types of elements, therefore we will have four different cases. Since each can verified directly by a simple computation, we will show the details for only one:

1.
$$\imath(e_{A,i}e_{A,j}) = \imath(e_{A,j})\imath(e_{A,i})$$

2.
$$\begin{aligned}
\imath(e_{A,i}(e_{A,j}e_{B,1}e_{A,k})) &= \imath(e_{A,j}e_{B,1}e_{A,k})\imath(e_{A,i}) \\
\imath((e_{A,i}e_{B,1}e_{A,j})e_{A,k}) &= \imath(e_{A,k})\imath(e_{A,j}e_{B,1}e_{A,k})
\end{aligned}$$

3. For $i \neq k$,
$$\imath((e_{A,i}e_{B,1}e_{A,j})(e_{A,k}e_{B,1}e_{A,l})) = (e_{A,l}e_{B,1}e_{A,k})(e_{A,j}e_{B,1}e_{A,i})$$

4.

$$\begin{aligned}
\imath((e_{A,i}e_{B,1}e_{A,j})(e_{A,j}e_{B,1}e_{A,k})) &= \imath(e_{A,i}(e_{B,1}e_{A,j}e_{B,1})e_{A,k}) \\
&= \imath(e_{A,i}(\mu_{j1}^{(d)}e_{B,1})e_{A,k}) \\
&= \overline{\mu_{j1}^{(d)}}\imath(e_{A,i}e_{B,1}e_{A,k}) \\
&= \mu_{j1}^{(d)}e_{A,k}e_{B,1}e_{A,i}
\end{aligned}$$

Also

$$\begin{aligned}
\imath(e_{A,j}e_{B,1}e_{A,k})\imath(e_{A,i}e_{B,1}e_{A,j}) &= (e_{A,k}e_{B,1}e_{A,j})(e_{A,j}e_{B,1}e_{A,i}) \\
&= e_{A,k}(e_{B,1}e_{A,j}e_{B,1})e_{A,i} \\
&= \mu_{j1}^{(d)}e_{A,k}e_{B,1}e_{A,i}
\end{aligned}$$

That $\imath^2 = \mathrm{Id}_{\mathcal{B}}$ follows immediately from the definition. □

Lemma 3.4. $\imath(e_{B,1}) = e_{B,1}$.

Proof: First note that $\imath(e_{A,i}e_{B,1}e_{A,i}) = \imath(\mu_{ii}^{(d)}e_{A,i}) = \mu_{ii}^{(d)}e_{A,i} = e_{A,i}e_{B,1}e_{A,i}$. Multiply $e_{B,1}$ by $1 = \sum_{i=1}^{d} e_{A,i}$ on both sides:

$$e_{B,1} = \left(\sum_{i=1}^{d} e_{A,i}\right) e_{B,1} \left(\sum_{j=1}^{d} e_{A,j}\right) = \sum_{i,j} e_{A,i}e_{B,1}e_{A,j}$$

into

$$\begin{aligned}
\imath(e_{B,1}) &= \imath\left(\sum_{i=1}^{d}\sum_{j=1}^{d} e_{A,i}e_{B,1}e_{A,j}\right) = \sum_{i=1}^{d}\sum_{j=1}^{d} \imath(e_{A,i}e_{B,1}e_{A,j}) \\
&= \sum_{i=1}^{d}\sum_{j=1}^{d}(e_{A,j}e_{B,1}e_{A,i}) = e_{B,1}
\end{aligned}$$

□

Corollary 3.5. *$\imath(A) = A^{-1}$, and $\imath(I) = I$.*

Proof:

$$\imath(A) = \imath(\sum_{i=1}^{d} \lambda_i e_{A,i}) = \sum_{i=1}^{d} \overline{\lambda_i}\imath(e_{A,i}) = \sum_{i=1}^{d} \lambda_i^{-1} e_{A,i} = A^{-1}$$

Similarly,

$$\imath(I) = \imath(\sum_{i=1}^{d} e_{A,i}) = \sum_{i=1}^{d} \imath(e_{A,i}) = \sum_{i=1}^{d} e_{A,i} = I$$

$\square$

Lemma 3.6. $\imath(B) = B^{-1}$.

Proof: Note that $A^{-1}\imath(B)A^{-1} = \imath(A)\imath(B)\imath(A) = \imath(ABA) = \imath(BAB) = \imath(B)A^{-1}\imath(B)$. That is A^{-1} and $\imath(B)$ satisfy the braid relation. So the group homomorphism $\rho' : B_3 \to \mathrm{GL}(V)$ defined by $\rho'(\sigma_1) = A^{-1}$ and $\rho'(\sigma_2) = \imath(B)$ is another representation of B_3 on V. Once again, the braid relation implies that A^{-1} and $\imath(B)$ are conjugates. Hence they have the same eigenvalues, namely $\lambda_1^{-1}, \lambda_2^{-1}, \dots, \lambda_d^{-1}$.

But $\imath : \mathcal{B} \to \mathcal{B}$ only permutes the basis S of $\mathcal{B} = \mathrm{End}(V)$. Hence $\imath(\mathcal{B}) = \imath(\mathrm{End}(V)) = \mathrm{End}(V)$ and A^{-1} and $\imath(B)$ generate the algebra $\mathrm{End}(V)$. That is ρ' is also a simple representation of B_3

Now, $(A^{-1}\imath(B))^3 = \imath(BA)^3 = \imath(AB)^3 = \imath(\delta I) = \overline{\delta} = \delta^{-1}I$ (recall $|\delta| = 1$). By Corollary 2.22, the eigenvalues $\lambda_1^{-1}, \lambda_2^{-1}, \dots, \lambda_d^{-1}$ (if d=2, 3) or the eigenvalues together with δ (if $d = 4, 5$) uniquely determine a simple representation of B_3 on V up to isomorphism.

But we already know such a representation, namely $\sigma_1 \mapsto A^{-1}$ and $\sigma_2 \mapsto B^{-1}$. Hence there exists $M \in \mathrm{GL}(V)$ such that $A^{-1} = MA^{-1}M^{-1}$ and $\imath(B) = MB^{-1}M^{-1}$. Then M is in the centralizer of A.

$$\begin{aligned}
Me_{B,1}M^{-1} &= M\left(\prod_{i=2}^{d} \frac{B - \lambda_i}{\lambda_1 - \lambda_i}\right)M^{-1} \\
&= \prod_{i=2}^{d} \frac{MBM^{-1} - \lambda_i}{\lambda_1 - \lambda_i} \\
&= \prod_{i=2}^{d} \frac{\imath(B^{-1}) - \lambda_i}{\lambda_1 - \lambda_i} \\
&= \prod_{i=2}^{d} \imath\left(\frac{B^{-1} - \lambda_i^{-1}}{\lambda_1^{-1} - \lambda_i^{-1}}\right) \\
&= \imath\left(\prod_{i=2}^{d} \frac{B^{-1} - \lambda_i^{-1}}{\lambda_1^{-1} - \lambda_i^{-1}}\right)
\end{aligned}$$

Call the quantity in parentheses ϕ. Note that ϕ is the eigenprojection to the subspace spanned by the eigenvector w_1 of B^{-1} with eigenvalue λ_1^{-1}. But the eigenvectors $w_1, w_2, \ldots, w_d$ of B^{-1} are also eigenvectors of B and span V (the eigenvalues are distinct). Hence $\phi(w_1) = w_1 = e_{B,1}w_1$ and $\phi(w_i) = 0 = e_{B,1}w_i$ for $i \geq 2$. That is $\phi = e_{B,1}$ as their action on the basis $\{w_1, w_2, \ldots, w_d\}$ is identical. Then Lemma 3.4 shows $\imath(Me_{B,1}M^{-1}) = \imath(\phi) = \imath(e_{B,1}) = e_{B,1}$.

Hence conjugation by M is a $\mathcal{B}$-algebra isomorphism that fixes A and $e_{B,1}$. But A and $e_{B,1}$ generate the basis S of $\mathcal{B}$, hence they generate the algebra $\mathcal{B}$. So conjugation by M must fix every element of $\mathcal{B}$. In particular, $\imath(B) = MB^{-1}M^{-1} = B^{-1}$. □

3.3 An Invariant Inner Product

Let $\mathcal{B}$ act on the left algebra ideal $\mathcal{B}e_{B,1}$. Note that $\mathcal{B}e_{B,1}$ is a d-dimensional $\mathbb{C}$-vector space, as $e_{B,1}$ is an idempotent of rank 1.

Definition 3.7. Define the form $\langle .\, , . \rangle$ on $\mathcal{B}e_{B,1}$ by $\langle ae_{B,1}, be_{B,1} \rangle \, e_{B,1} = \imath(be_{B,1})ae_{B,1} = e_{B,1}\imath(b)ae_{B,1}$ for $ae_{B,1}, be_{B,1} \in \mathcal{B}e_{B,1}$.

It is easy to verify that $\langle .\, , . \rangle$ is a sesquilinear form on the $\mathbb{C}$-vector space $\mathcal{B}e_{B,1}$. Since $\imath(A) = A^{-1}$ and $\imath(B) = B^{-1}$, this form is clearly invariant under the action by A and B, hence $\rho(B_3)$.

Lemma 3.8. $T = \{e_{A,i}e_{B,1} \mid 2 \leq i \leq d\} \cup \{ABA\, e_{B,1}\}$ *is a basis for the left algebra ideal* $\mathcal{B}e_{B,1}$ *considered as a* $\mathbb{C}$*-vector space.*

Proof: Suppose

$$\alpha_1 ABA\, e_{B,1} + \sum_{i=2}^{d} \alpha_i e_{A,i}e_{B,1} = 0$$

Note that $(e_{A,i}ABA\, e_{B,1})(ABA)^{-1} = e_{A,i}e_{A,1} = \delta_{1i}\, e_{A,1}$. Since $(ABA)^{-1}$ is invertible $e_{A,i}ABA\, e_{B,1} = 0$ if and only if $i \geq 2$.

Multiply by $e_{A,1}$ on the left. Then $\alpha_1 e_{A,1}ABA\, e_{B,1} = 0$ But $e_{A,1}ABA\, e_{B,1} \neq 0$, so $\alpha_1 = 0$.

Now, multiply by $e_{A,i}$ $(i \geq 2)$ on the left. Then $\alpha_i e_{A,i}e_{B,1} = 0$. We know $e_{B,1}e_{A,i}e_{B,1} = \mu_{i1}^{(d)} e_{B,1} \neq 0$ by simplicity, so $e_{A,i}e_{B,1} \neq 0$ and $\alpha_i = 0$.

Hence T is a linearly independent set, and we can conclude that it is a basis of the d-dimensional vector space $\mathcal{B}e_{B,1}$. □

Note: if we know $e_{A,1}e_{B,1} \neq 0$, we can use the more symmetric basis $T' = \{e_{A,i}e_{B,1} \mid 1 \leq i \leq d\}$ to simplify this and some of the following computations. Unfortunately, $e_{A,1}e_{B,1}$ could in general be 0. In particular, if $\mu_{11}^{(d)} = 0$, then $e_{A,1}e_{B,1} = 0$ too.

Theorem 3.9. *The braid representation $\mathcal{B}$ is unitarizable if and only if $\mu_{i1}^{(d)} > 0$ for all $2 \leq i \leq d$.*

Proof: Suppose $\mu_{i1}^{(d)} > 0$ for all $2 \leq i \leq d$. Consider the action of $\mathcal{B}$ on $\mathcal{B}e_{B,1}$. The sesquilinear form defined above is invariant under the action of $\rho(B_3)$. So it is sufficient to show that it is an inner product. That is we need to prove that it is positive definite. On the basis T:

$$
\begin{aligned}
\langle e_{A,i}e_{B,1}, e_{A,i}e_{B,1}\rangle\, e_{B,1} &= e_{B,1}\imath(e_{A,i})e_{A,i}e_{B,1} = e_{B,1}e_{A,i}e_{A,i}e_{B,1} \\
&= e_{B,1}e_{A,i}e_{B,1} = \mu_{i1}^{(d)}e_{B,1} \\
\langle ABA\, e_{B,1}, ABA\, e_{B,1}\rangle\, e_{B,1} &= \langle e_{B,1}, e_{B,1}\rangle\, e_{B,1} = e_{B,1}e_{B,1} = e_{B,1}
\end{aligned}
$$

Hence $\langle e_{A,i}e_{B,1}, e_{A,i}e_{B,1}\rangle = \mu_{i1}^{(d)}$ for $i \geq 2$, which is positive by our condition, and $\langle ABA\, e_{B,1},\, ABA\, e_{B,1}\rangle = 1$. We claim that T is orthogonal with respect to $\langle ., , \rangle$. Let $i, j \neq 1$ and $i \neq j$:

$$
\begin{aligned}
\langle e_{A,i}e_{B,1}, e_{A,j}e_{B,1}\rangle\, e_{B,1} &= e_{B,1}\imath(e_{A,i})e_{A,j}e_{B,1} = e_{B,1}e_{A,i}e_{A,j}e_{B,1} = 0 \\
\langle ABA\, e_{B,1}, e_{A,i}e_{B,1}\rangle\, e_{B,1} &= e_{B,1}\imath(e_{A,i})ABA\, e_{B,1} = e_{B,1}e_{A,i}ABA\, e_{B,1} = 0
\end{aligned}
$$

We used $e_{A,i}ABA\, e_{B,1} = 0$ in the last computation just like in Lemma 3.8.

Hence $\langle ., .\rangle$ is a positive definite form. Then $\mathcal{B}e_{B,1}$ is a $\mathbb{C}$-vector space with inner product $\langle . , .\rangle$ and the action of $\rho(B_3)$ on this space is unitary.

Conversely, suppose $\mathcal{B}$ is unitarizable. So there exists V a $\mathbb{C}$ vector space with inner product $\langle . , .\rangle$ and $\rho : B_3 \to \mathrm{GL}(V)$ such that $A = \rho(\sigma_1)$ and $B = \rho(\sigma_2)$ act as unitary operators on V. Let * be the transpose induced by $\langle . , .\rangle$. We know $A^* = A^{-1}$ and $B^* = B^{-1}$. Let $v \in V$ be an eigenvector of B with eigenvalue λ_1. Then $e_{B,1}v = v$

and

$$\begin{aligned}
0 \leq \langle e_{A,i} e_{B,1} v, e_{A,i} e_{B,1} v \rangle &= \left\langle v, e_{B,1}^* e_{A,i}^* e_{A,i} e_{B,1} v \right\rangle \\
&= \langle v, e_{B,1} e_{A,i} e_{B,1} v \rangle = \left\langle v, \mu_{i1}^{(d)} e_{B,1} v \right\rangle = \mu_{i1}^{(d)} \langle v, v \rangle
\end{aligned}$$

Hence $\mu_{i1}^{(d)} \geq 0$. We know $\mu_{i1}^{(d)} \neq 0$ for $i \geq 2$ by simplicity, so $\mu_{i1}^{(d)} > 0$ in this case. □

3.4 Examples

Example 3.10. $d = 2$

$$\begin{aligned}
\mu_{21}^{(2)} &= \frac{-\lambda_1^2 + \lambda_1\lambda_2 - \lambda_2^2}{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_1)} \\
&= \frac{\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2}{(\lambda_1 - \lambda_2)^2} \\
&= 1 + \frac{\lambda_1\lambda_2}{(\lambda_1 - \lambda_2)^2} \\
&= 1 - \frac{1}{(\lambda_1/\lambda_2 - 1)(\lambda_2/\lambda_1 - 1)} \\
&= 1 - \left| \frac{\lambda_1}{\lambda_2} - 1 \right|^{-2} > 0
\end{aligned}$$

That is

$$\left| \frac{\lambda_1}{\lambda_2} - 1 \right| > 1$$

or $\lambda_1/\lambda_2 = e^{it}$ for $\pi/3 < t < 5\pi/3$.

Example 3.11. $d = 3$

$$\begin{aligned}
\mu_{21}^{(3)} &= \frac{(\lambda_1^2 + \lambda_2\lambda_3)(\lambda_2^2 + \lambda_1\lambda_3)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \\
&= \frac{\left(1 + \frac{\lambda_3}{\lambda_1}\frac{\lambda_2}{\lambda_1}\right)\left(\frac{\lambda_2}{\lambda_1} + \frac{\lambda_3}{\lambda_2}\right)}{\left(1 - \frac{\lambda_2}{\lambda_1}\right)\left(1 - \frac{\lambda_1}{\lambda_2}\right)\left(1 - \frac{\lambda_3}{\lambda_1}\right)\left(\frac{\lambda_2}{\lambda_1} - \frac{\lambda_3}{\lambda_1}\right)} \\
\mu_{31}^{(3)} &= \frac{(\lambda_1^2 + \lambda_2\lambda_3)(\lambda_3^2 + \lambda_1\lambda_2)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \\
&= \frac{\left(1 + \frac{\lambda_2}{\lambda_1}\frac{\lambda_3}{\lambda_1}\right)\left(\frac{\lambda_3}{\lambda_1} + \frac{\lambda_2}{\lambda_3}\right)}{\left(1 - \frac{\lambda_3}{\lambda_1}\right)\left(1 - \frac{\lambda_1}{\lambda_3}\right)\left(1 - \frac{\lambda_2}{\lambda_1}\right)\left(\frac{\lambda_3}{\lambda_1} - \frac{\lambda_2}{\lambda_1}\right)}
\end{aligned}$$

Let $\omega_2 = \lambda_2/\lambda_1$ and $\omega_3 = \lambda_3/\lambda_1$. Then

$$\begin{aligned}
\mu_{21}^{(3)} &= \frac{(1+\omega_3\omega_2)\left(\omega_2+\omega_3\omega_2^{-1}\right)}{|1-\omega_2|^2\,(1-\omega_3)\,(\omega_2-\omega_3)} \\
\mu_{31}^{(3)} &= \frac{(1+\omega_2\omega_3)\left(\omega_3+\omega_2\omega_3^{-1}\right)}{|1-\omega_3|^2\,(1-\omega_2)\,(\omega_3-\omega_2)}
\end{aligned}$$

Let $e^{2\pi t_2} = \omega_2$ and $e^{2\pi t_3} = \omega_3$. So we are looking for $(t_2, t_3) \in [0,1)^2$ such that both $\mu_{21}^{(3)} > 0$ and $\mu_{31}^{(3)} > 0$. $\mu_{21}^{(3)}$ and $\mu_{31}^{(3)}$ can change signs at

$$\begin{aligned}
\omega_2\omega_3 &= -1 \\
\omega_3\omega_2^{-1} &= -\omega_2 \\
\omega_2\omega_3^{-1} &= -\omega_3 \\
w_2 &= 1 \\
w_3 &= 1 \\
w_2 &= w_3
\end{aligned}$$

These equations can be transformed into linear equations in t_2 and t_3 by taking logs:

$$\begin{aligned}
t_2 + t_3 &= \frac{1}{2} \\
t_3 &= 2t_2 + \frac{1}{2} \\
t_2 &= 2t_3 + \frac{1}{2} \\
t_2 &= 0 \\
t_3 &= 0 \\
t_2 &= t_3
\end{aligned}$$

Of course, the above equations are all understood mod 1.

Computation by Maple shows that $\mu_{21}^{(3)} > 0$ and $\mu_{31}^{(3)} > 0$ in the open set colored black on the plot below. The grey regions are those where one of $\mu_{21}^{(3)}$ and $\mu_{31}^{(3)}$ is positive and the other is negative. The line $t_2 = t_3$ corresponds to $\lambda_2 = \lambda_3$, in which case the representation cannot be unitarizable.

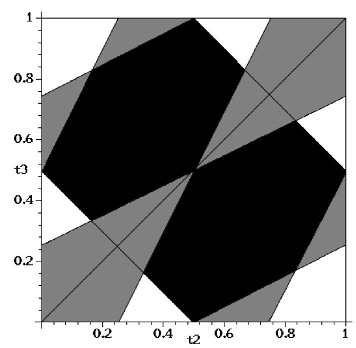


Figure 3.1: Region of unitarizability for $d = 3$

Chapter 4

Tensor Categories

4.1 Definitions

Definition 4.1. A monoidal category $\tilde{\mathcal{C}}$ is a category $\mathcal{C}$ with a tensor product $\otimes : \tilde{\mathcal{C}} \times \tilde{\mathcal{C}} \to \tilde{\mathcal{C}}$ on the objects and morphisms of $\tilde{\mathcal{C}}$, a natural transformation a between $\otimes \circ (\otimes \times \mathrm{Id}_{\tilde{\mathcal{C}}})$ and $\otimes \circ (\mathrm{Id}_{\tilde{\mathcal{C}}} \times \otimes)$, and a unit object $\mathbb{1} \in \tilde{\mathcal{C}}$ such that

1.
$$\begin{tikzcd}
((X \otimes Y) \otimes V) \otimes W \arrow[r, "a_{X,Y,V} \otimes \mathrm{Id}_W"] \arrow[d, "a_{X \otimes Y, V, W}"] & (X \otimes (Y \otimes V)) \otimes W \arrow[dd, "a_{X, Y \otimes V, W}"] \\
(X \otimes Y) \otimes (V \otimes W) \arrow[d, "a_{X,Y,V \otimes W}"] & \\
X \otimes (Y \otimes (V \otimes W)) & X \otimes ((Y \otimes V) \otimes W) \arrow[l, "\mathrm{Id}_X \otimes a_{Y,V,W}"]
\end{tikzcd}$$

   is commutative.

2. $X \otimes \mathbb{1} \cong \mathbb{1} \otimes X \cong X$ for all $X \in \tilde{\mathcal{C}}$.

Definition 4.2. A monoidal category is called strict if a is the identity and $\mathbb{1} \otimes X = X \otimes \mathbb{1} = X$ for any $X \in \tilde{\mathcal{C}}$.

Definition 4.3. A strict monoidal category is called rigid if every object $X \in \tilde{\mathcal{C}}$ has a dual object $X^* \in \tilde{\mathcal{C}}$ and a pair of morphisms $i_X : \mathbb{1} \to X \otimes X^*$ and $e_X : X^* \otimes X \to \mathbb{1}$ such that the maps

$$X = \mathbb{1} \otimes X \xrightarrow{i_X \otimes \mathrm{Id}_X} X \otimes X^* \otimes X \xrightarrow{\mathrm{Id}_X \otimes e_X} X \otimes \mathbb{1} = X$$

$$X^* = X^* \otimes \mathbb{1} \xrightarrow{\mathrm{Id}_X \otimes i_X} X^* \otimes X \otimes X^* \xrightarrow{e_X \otimes \mathrm{Id}_X} \mathbb{1} \otimes X^* = X^*$$

are Id_X and Id_{X^*}.

Definition 4.4. A tensor category is a monoidal category equipped with a direct sum $\oplus : \tilde{\mathcal{C}} \times \tilde{\mathcal{C}} \to \tilde{\mathcal{C}}$ and an operation of projection onto subobjects.

Definition 4.5. The Grothendieck semiring $\mathcal{R}$ of the tensor category $\tilde{\mathcal{C}}$ is the set of equivalence classes of objects of $\tilde{\mathcal{C}}$ with $\oplus$ and $\otimes$ as addition and multiplication.

We call an object X in the category simple if $\mathrm{End}(X)$ is a field. We will always assume that $\mathbb{1}$ is simple. A tensor category is semisimple if each object is semisimple, that is it is a direct sum of simple objects.

Definition 4.6. A monoidal category $\tilde{\mathcal{C}}$ is called braided if there exists a family c of natural isomorphisms $c_{V,W} : V \otimes W \to W \otimes V$ such that:

$$\begin{tikzcd} X \otimes Y \otimes Z \arrow[rr, "c_{X,Y\otimes Z}"'] \arrow[dr, "c_{X,Y}\otimes \mathrm{Id}_Z"'] & & X \otimes Z \otimes Y \\ & Y \otimes X \otimes Z \arrow[ur, "\mathrm{Id}_Y \otimes c_{X,Z}"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} X \otimes Y \otimes Z \arrow[rr, "c_{X\otimes Y,Z}"'] \arrow[dr, "\mathrm{Id}_U \otimes c_{Y,Z}"'] & & Y \otimes Z \otimes X \\ & X \otimes Z \otimes Y \arrow[ur, "c_{X,Z}\otimes \mathrm{Id}_Y"'] & \end{tikzcd}$$

commute. Naturality means that for any morphisms $f : X \to X'$ and $g : Y \to Y'$

$$(f \otimes g) \circ c_{X,Y} = c_{X',Y'} \circ (f \otimes g).$$

This is a generalization of the flip, which is the natural isomorphism between $P_{A,B} : A \otimes B \to B \otimes A$, where A and B are modules over the commutative ring R. Note that the flip is involutive, that is $P_{B,A} \circ P_{A,B} = \mathrm{Id}_{A\otimes B}$. This is not required for a braiding, but the property is generalized in the notion of a twist:

Definition 4.7. A twist in a braided monoidal category $\tilde{\mathcal{C}}$ is family θ of isomorphisms $\theta_V : V \to V$ such that

$$\theta_{X\otimes Y} = c_{Y,X} \circ c_{X,Y} \circ (\theta_X \otimes \theta_Y)$$

for all $X, Y \in \tilde{\mathcal{C}}$. θ is required to be natural in the sense that for any morphism $f : X \to Y$, $\theta_Y \circ f = f \circ \theta_X$.

Definition 4.8. A ribbon category $\tilde{\mathcal{C}}$ is a rigid braided monoidal category with a compatible twist, meaning:

$$(\theta_X \otimes \mathrm{Id}_{X^*}) \circ i_X = (\mathrm{Id}_X \otimes \theta_{X^*}) \circ i_X.$$

In a ribbon category, we can define the trace of an endomorphism and the dimension of an object as follows.

Definition 4.9. Let $\tilde{\mathcal{C}}$ be a ribbon category, $X \in \tilde{\mathcal{C}}$, and $f \in \mathrm{End}(X)$. Then the trace of f is defined as

$$\mathrm{tr}(f) = e_X \circ c_{X,X^*} \circ ((\theta_X \circ f) \otimes \mathrm{Id}_{X^*}) \circ i_X \in \mathrm{End}(\mathbb{1})$$

and the categorical dimension of X as

$$\dim X = \mathrm{tr}(\mathrm{Id}_X).$$

It can be shown (see [13]) that $\mathrm{tr}(fg) = \mathrm{tr}(gf)$ for any $f \in \mathrm{Hom}(X,Y)$ and $g \in \mathrm{Hom}(Y,X)$. Also $\mathrm{tr}(f \otimes g) = \mathrm{tr}(f)\,\mathrm{tr}(g)$ for any $f \in \mathrm{End}(X)$ and $g \in \mathrm{End}(Y)$. If $f \in \mathrm{End}(\mathbb{1})$, then $\mathrm{tr}(f) = f$.

4.2 An Application of Braid Representations

Let $\tilde{\mathcal{C}}$ be a semisimple ribbon tensor category with $\mathrm{End}(\mathbb{1}) = F$ an algebraically closed field. Then $\mathrm{Hom}(X,Y)$ is an F-vector space for all $X, Y \in \tilde{\mathcal{C}}$ and $\mathrm{End}(X)$ is a semisimple F-algebra.

Assume that $\tilde{\mathcal{C}}$ contains a self dual object Z, that is $\mathbb{1}$ appears exactly once in the direct sum decomposition of $Z \otimes Z$. Let $p \in \mathrm{End}(Z \otimes Z)$ be the projection to $\mathbb{1}$ and $p^{(1)} = p \otimes \mathrm{Id}_Z$ and $p^{(2)} = \mathrm{Id}_Z \otimes p$ in $\mathrm{End}(Z^{\otimes 3})$.

Lemma 4.10.

$$p^{(2)}\,p^{(1)}\,p^{(2)} \neq 0.$$

Proof: Let $q = i_Z\, e_Z \in \mathrm{End}(Z^{\otimes 2})$, $q^{(1)} = q \otimes \mathrm{Id}_Z$, and $q^{(2)} = \mathrm{Id}_Z \otimes q$. Then

$$q^{(2)}\, q^{(1)}\, q^{(2)} = (\mathrm{Id}_Z \otimes i_Z)\, \underbrace{(\mathrm{Id}_Z \otimes e_Z)(i_Z \otimes \mathrm{Id}_Z)}_{\mathrm{Id}_Z}\, \underbrace{(e_Z \otimes \mathrm{Id}_Z)(\mathrm{Id}_Z \otimes i_Z)}_{\mathrm{Id}_Z}(\mathrm{Id}_Z \otimes e_Z) = q^{(2)}.$$

Note $q^{(1)} \neq 0$ because

$$\mathrm{Id}_Z = \underbrace{(\mathrm{Id}_Z \otimes e_Z)(i_Z \otimes \mathrm{Id}_Z)}_{\mathrm{Id}_Z}\, \underbrace{(e_Z \otimes \mathrm{Id}_Z)(\mathrm{Id}_Z \otimes i_Z)}_{\mathrm{Id}_Z} = (\mathrm{Id}_Z \otimes e_Z) q^{(1)} (\mathrm{Id}_Z \otimes i_Z)$$

and similarly $q^{(2)} \neq 0$ either. Let $\alpha = e_Z\, i_Z \in \mathrm{End}(\mathbb{1}) = F$. Then

$$q^2 = i_Z\, \underbrace{e_Z\, i_Z}_{\alpha}\, e_Z = \alpha\, q.$$

and hence $q_i^2 = \alpha\, q_i$ for $i = 1,\, 2$.

Observe that $q \in \mathrm{End}(Z^{\otimes 2})$ which is a semisimple F-algebra, hence isomorphic to a direct sum of full matrix rings. Suppose $\alpha = 0$ hence q is nilpotent. As q can only be nonzero on the direct summand $\mathbb{1}$ of $Z^{\otimes 2}$ and the multiplicity of $\mathbb{1}$ is 1, q is nilpotent if and only if $q = 0$. But then $q^{(1)} = 0$, which we have proven is not the case. Therefore $\alpha \neq 0$.

Note $(1/\alpha\, q^{(i)})^2 = 1/\alpha\, q^{(i)}$ for $i = 1,\, 2$ and $\mathrm{im}\, 1/\alpha\, q^{(i)} = \mathrm{im}\, p_i$. Hence

$$p^{(2)}\, p^{(1)}\, p^{(2)} = \frac{1}{\alpha^3}\, q^{(2)}\, q^{(1)}\, q^{(2)} = \frac{1}{\alpha^3}\, q^{(2)} \neq 0.$$

□

Let $f \in \mathrm{End}(Z^{\otimes 2})$. Note that $Z \otimes \mathbb{1} = Z$, hence $p^{(2)}(f \otimes \mathrm{Id}_Z)p^{(2)}$ is a multiple of $p^{(2)}$. In particular, let $f = p_X$ be a projection onto some term X in a direct sum decomposition of $Z^{\otimes 2}$. Define $\dim X$ by

$$(\dim X)p^{(2)} = (\dim Z)^2 p^{(2)}(p_X \otimes \mathrm{Id}_Z)p^{(2)}$$

where we choose $\dim Z$ so that $\dim \mathbb{1} = 1$. This determines $\dim Z$ up to sign and $\dim X$ is clearly independent of the choice of sign. It can be checked that this definition of $\dim X$ is equivalent to the usual one given in Definition 4.9 for direct summands in $Z^{\otimes 2}$.

Let $c_1 = c_{Z,Z} \otimes \mathrm{Id}_Z$ and $c_2 = \mathrm{Id}_Z \otimes c_{Z,Z}$. By the definition of braiding $c_1\, c_2\, c_1 = c_2\, c_1\, c_2$. Assume $Z^{\otimes 2} = \bigoplus_i X_i$ where the X_i are d nonisomorphic simple objects of nonzero dimension. Then the braiding $c_{Z,Z}$ acts on these simple objects via scalars λ_i. Assume that the λ_i are distinct.

Proposition 4.11. *In this case, we can define an action of B_3 on $V = \mathrm{Hom}(Z, Z^{\otimes 3})$ by $\sigma_i f = c_i \circ f$ for $f \in \mathrm{Hom}(Z, Z^{\otimes 3})$. Then V is a simple B_3 module and each eigenvalue of σ_i is of multiplicity* 1.

Proof: Index the X_i so that $X_1 = \mathbb{1} \subseteq Z^{\otimes 2}$. Then $p^{(1)} = p_{X_1} \otimes \mathrm{Id}_Z$ and $p^{(2)} = \mathrm{Id}_Z \otimes p_{X_1}$. Let $\imath : \mathbb{1} \to Z^{\otimes 2}$ be a nonzero morphism. Then $\mathrm{im}\,\imath = x_1$. As $\dim X_i \neq 0$, the projections $p^{(i)}$ must be nonzero when restricted to $Z \otimes X_1 \subseteq Z^{\otimes 3}$. Hence $v_i = (p^{(i)} \otimes \mathrm{Id}_Z)(\mathrm{Id}_Z \otimes \imath) \neq 0$ and $\sigma_1 v_i = \lambda_i v_i$. As $\dim \mathrm{Hom}(Z, Z^{\otimes 3}) = \dim \mathrm{Hom}(Z^{\otimes 2}, Z^{\otimes 2})$, the v_i form an eigenbasis of V for c_1.

Suppose V is not simple. Let $0 \subseteq V_1 \subseteq \ldots \subseteq V_n = V$ be a composition series of V. Clearly, each $p^{(i)} \otimes \mathrm{Id}_Z$ acts nonzero on exactly one simple factor in the series, and each simple factor has at least one $p^{(i)} \otimes \mathrm{Id}_Z$ acting nonzero on it. There are at least two simple factors so we can choose i so that $p^{(i)} \otimes \mathrm{Id}_Z$ and $p^{(1)}$ act nonzero on different simple factors. Since $p^{(2)}$ is conjugate to $p^{(1)}$, $p^{(2)}$ acts nonzero on the same simple factor as $p^{(1)}$, Hence $p^{(2)}(p^{(i)} \otimes \mathrm{Id}_Z)p^{(2)} = 0$ which would contradict $\dim x_i \neq 0$. □

Corollary 4.12. *We have*

$$\dim X_i = \mu_{i1}^{(d)} (\dim Z)^2$$

with $\mu_{i1}^{(d)}$ as in Chapter 3.

Proof: As the eigenvalues are all of multiplicity 1, we have well-defined eigenprojections $p^{(i)} = P_i^{(d)}(c_1)/P_i^{(d)}(\lambda_i)$ and $p^{(2)} = P_1^{(d)}(c_2)/P_1^{(d)}(\lambda_1)$. Hence

$$\begin{aligned}
(\dim X_i)\frac{P_1^{(d)}(c_2)}{P_1^{(d)}(\lambda_1)} &= (\dim Z)^2 \frac{P_1^{(d)}(c_2)P_i^{(d)}(c_1)P_1^{(d)}(c_2)}{P_1^{(d)}(\lambda_1)P_i^{(d)}(\lambda_i)P_1^{(d)}(\lambda_1)} \\
&= (\dim Z)^2 \frac{Q_{i1}^{(d)}P_1^{(d)}(c_2)}{P_1^{(d)}(\lambda_1)P_i^{(d)}(\lambda_i)P_1^{(d)}(\lambda_1)}.
\end{aligned}$$

□

In particular, let $\mathcal{C}$ be a braided tensor category whose Grothendieck semiring is isomorphic to that of the representation category of $\mathfrak{g}$ where $\mathfrak{g}$ is of orthogonal or symplectic type. Let $Z \in \mathcal{C}$ be the object corresponding to the vector representation of $\mathfrak{g}$. Then $Z \otimes Z \cong \mathbb{1} \otimes X \otimes Y$ and the above result applies with $d = 3$. Choose $alpha \in \mathbb{C}$ so that the eigenvalues of c_1 on X and Y are αq and $-\alpha q^{-1}$. Denote the eigenvalue on

$\mathbb{1}$ by r^{-1}. It can be shown that α is a fourth root of 1, but we will not need it in this discussion. The categorical dimensions are

$$\dim X = \left(\frac{rq - r^{-1}q^{-1}}{q^2 - q^{-2}} + 1\right)\frac{r - r^{-1}}{q - q^{-1}}$$

$$\dim Y = \left(\frac{rq^{-1} - r^{-1}q}{q^2 - q^{-2}} + 1\right)\frac{r - r^{-1}}{q - q^{-1}}$$

$$\dim Z = \pm\left(\frac{r - r^{-1}}{q - q^{-1}} + 1\right).$$

It is possible to prove that $r = q^{N-1}$ for $\mathfrak{g}$ is orthogonal type and $r = q^{-N-1}$ if it is symplectic.

If $\mathfrak{g}$ is an exceptional Lie algebra, we can choose Z to correspond to the adjoint representation to get a 5-dimensional simple representation of B_3. The categorical dimensions of the 5 simple summands of $\mathbb{Z} \otimes Z$ can be computed like in the previous case (see [11]).

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