LOW-DIMENSIONAL UNITARY REPRESENTATIONS OF B_3

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ABSTRACT. We characterize all simple unitarizable representations of the braid group B_3 on complex vector spaces of dimension $d \leq 5$. In particular, we prove that if σ_1 and σ_2 denote the two generating twists of B_3, then a simple representation $\rho : B_3 \to \mathrm{GL}(V)$ (for $\dim V \leq 5$) is unitarizable if and only if the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_d$ of $\rho(\sigma_1)$ are distinct, satisfy $|\lambda_i| = 1$ and $\mu_{1i}^{(d)} > 0$ for $2 \leq i \leq d$, where the $\mu_{1i}^{(d)}$ are functions of the eigenvalues, explicitly described in this paper.

1. Introduction

Unitary braid representations have been constructed in several ways using the representation theory of Kac-Moody algebras and quantum groups, (see e.g. [1], [2], and [4]), and specializations of the reduced Burau and Gassner representations in [5]. Such representations easily lead to representations of $\mathrm{PSL}(2, \mathbb{Z}) = B_3/Z$, where Z is the center of B_3, and $\mathrm{PSL}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})/\{\pm 1\}$, where $\{\pm 1\}$ is the center of $\mathrm{SL}(2, \mathbb{Z})$. We give a complete classification of simple unitary representations of B_3 of dimension $d \leq 5$ in this paper. In particular, the unitarizability of a braid representation depends only on the the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_d$ of the images of the two generating twists of B_3. The condition for unitarizability is a set of linear inequalities in the logarithms of these eigenvalues. In other words, the representation is unitarizable if and only if the $(\arg \lambda_1, \arg \lambda_2, \ldots, \arg \lambda_d)$ is a point inside a polyhedron in $(\mathbb{R}/2\pi)^d$, where we give the equations of the hyperplanes that bound this polyhedron. This classification shows that the approaches mentioned previously do not produce all possible unitary braid representations. We obtain representations that seem to be new for $d \geq 3$. As any unitary representation of B_n restricts to a unitary representation of B_3 in an obvious way, these results may also be useful in classifying such representation of B_n.

I would like to thank my advisor, Hans Wenzl, for the ideas he contributed to this paper and Nolan Wallach for his suggestions.

Let B_3 be generated by σ_1 and σ_2 with the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. It is well-known that the center of B_3 is generated by $(\sigma_1\sigma_2)^3$. Let K be any field. If ρ is a simple representation of B_3 on a K-vector space V, then $\rho(\sigma_1\sigma_2)^3$ must act on V as a scalar $\delta \in K$. Since σ_1 and σ_2 are conjugates via $\sigma_1\sigma_2\sigma_1$, their images $A = \rho(\sigma_1)$ and $B = \rho(\sigma_2)$ have the same eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_d$. We will need the following two results from [3].

Theorem 1.1. (1) *Let K be an algebraically closed field, V a d-dimensional K-vector space, and $\lambda_1, \lambda_2, \ldots, \lambda_d \in K - \{0\}$, where $d \leq 5$. There exists*

Date: January 31, 2000.

1991 *Mathematics Subject Classification.* Primary 20F36, 20C07, 81R10; Secondary 20H20, 16S34.

a simple representation $\rho : B_3 \to \mathrm{GL}(V)$ *such that the eigenvalues of* $A = \rho(\sigma_1)$ *satisfy* $Q_{rs}^{(d)} \neq 0$ *for all* $r \neq s$ *where the polynomials* $Q_{rs}^{(d)}$ *are as follows:*

$$Q_{rs}^{(2)} = -\lambda_r^2 + \lambda_r\lambda_s - \lambda_s^2,$$
$$Q_{rs}^{(3)} = (\lambda_r^2 + \lambda_s\lambda_k)(\lambda_s^2 + \lambda_r\lambda_k)$$

with $k \neq r, s$.

$$Q_{rs}^{(4)} = -\gamma^{-1}(\lambda_r^2 + \gamma)(\lambda_s^2 + \gamma)(\gamma + \lambda_r\lambda_k + \lambda_s\lambda_l)(\gamma + \lambda_r\lambda_l + \lambda_s\lambda_k)$$

with $\gamma^2 = \lambda_1 \cdots \lambda_4$ *and* $k, l \neq r, s$.

$$Q_{rs}^{(5)} = \gamma^{-8}(\gamma^2 + \lambda_r\gamma + \lambda_r^2)(\gamma^2 + \lambda_s\gamma + \lambda_s^2)\prod_{k\neq r,s}(\gamma^2 + \lambda_r\lambda_k)(\gamma^2 + \lambda_s\lambda_k)$$

with $\gamma^5 = \lambda_1 \cdots \lambda_5$.

(2) *A simple representation of* B_3 *of dimension* $d \leq 5$ *is uniquely determined up to isomorphism by the eigenvalues of* $A = \rho(\sigma_1)$ *(for* $d \leq 3$*) and* δ*, where* $\rho(\sigma_1\sigma_2)^3 = \delta\, \mathrm{Id}_V$ *(for* $d = 4, 5$*).*

Explicit matrices for $A = \rho(\sigma_1)$ and $B = \rho(\sigma_2)$ are also listed in [3].

The functions $Q_{rs}^{(d)}$ are defined in [3] by

$$P_r^{(d)}(B)P_s^{(d)}(A)P_r^{(d)}(B) = Q_{rs}^{(d)}P_r^{(d)}(B),$$

where $P_r^{(d)}(x) = \prod_{i\neq r}(x - \lambda_i)$. Note that substituting $\lambda_i = e^{2\pi i t_i}$ and taking logarithms reduces the problem of finding the zeroes of $Q_{rs}^{(d)}$ to solving a system of linear equations in the t_i. (See Example 4.2.)

Proposition 1.2. *Let* $\rho : B_3 \to \mathrm{GL}(V)$ *be a simple representation of dimension* $d \leq 5$. *Then the minimal polynomials of* $A = \rho(\sigma_1)$ *and* $B = \rho(\sigma_2)$ *are the same as their characteristic polynomials.*

An immediate consequence of this is

Corollary 1.3. *If* A *(or* B*) is a diagonalizable matrix, then it has distinct eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_d$.

Proof. Since A is conjugate to some diagonal matrix D, its minimal polynomial is just $p(x) = \prod(x - d_j)$ where the d_j are the distinct diagonal entries of D. By the previous proposition, $\deg p = d$; hence D must have d distinct diagonal entries. Thus all of the diagonal entries of D are distinct. □

Since we are interested in unitarizable representations, we will let $K = \mathbb{C}$ and we will require that $|\lambda_i| = 1$. Let $\rho : B_3 \to V$ be a simple d-dimensional representation ($d \leq 5$), and $A = \rho(\sigma_1)$, $B = \rho(\sigma_2)$. Any unitarizable complex matrix is diagonalizable, so we can assume that A and B are diagonalizable. So the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_d$ are distinct by the last corollary. Let δ be the scalar via which $\rho(\sigma_1\sigma_2)^3$ acts on V, that is $(AB)^3 = \delta I$. Denote the $\mathbb{C}$-algebra generated by A and B by $\mathcal{B}$. In other words, $\mathcal{B} = \rho(\mathbb{C}\, B_3)$, where $\mathbb{C}\, B_3$ is the group algebra. Note that $\mathcal{B} = \mathrm{End}(V)$ by simplicity.

The proof proceeds by defining a vector space antihomomorphism $\imath : \mathcal{B} \to \mathcal{B}$ and proving that it is an algebra antihomorphism and an involution of $\mathcal{B}$ in section 2. In section 3, we define a sesquilinear form $\langle\, .\, , .\, \rangle$ on the ideal $I = \mathcal{B}e_{B,1}$ that is invariant under multiplication by A and B. We prove that $\langle\, .\, , .\, \rangle$ is positive definite if $\mu_{1i}^{(d)} > 0$ for $2 \le i \le d$. In this case, ρ is a unitary representation of B_3 on the d-dimensional vector space I. We also prove that ρ is a unitarizable representation $\mu_{1i}^{(d)} > 0$ for $2 \le i \le d$. In section 4, we give some examples of using the positivity of $\mu_{1i}^{(d)}$.

2. An involution of the image of B_3

Let $e_{M,i}$ be the eigenprojection of M to the eigenspace of λ_i, where $M \in \{A, B\}$. That is,

$$e_{M,i} = \prod_{j \neq i} \frac{M - \lambda_j}{\lambda_i - \lambda_j} = \frac{P_i^{(d)}(M)}{\prod_{j\neq i}(\lambda_i - \lambda_j)}.$$

Note that $e_{A,i}$ and $e_{B,i}$ always exist because the eigenvalues are distinct. Also $e_{M,i}e_{M,j} = \delta_{ij} e_{M,i}$. Define $\mu_{ij}^{(d)}$ by $e_{B,i}e_{A,j}e_{B,i} = \mu_{ij}^{(d)} e_{B,i}$. Note that

$$\mu_{ij}^{(d)} = \frac{Q_{ij}^{(d)}}{\prod_{k\neq i}(\lambda_i - \lambda_k)\prod_{k\neq j}(\lambda_j - \lambda_k)}.$$

Lemma 2.1. *The $\mu_{ij}^{(d)}$ are real numbers.*

Proof. For $i \neq j$, the proof is by direct computation using $\overline{\lambda_i} = \lambda_i^{-1}$ and $\overline{\gamma} = \gamma^{-1}$. For example, for $d = 5$:

$$\begin{aligned}
\mu_{ij}^{(d)} &= \frac{(\gamma^2 + \lambda_i\gamma + \lambda_i^2)(\gamma^2 + \lambda_j\gamma + \lambda_j^2)\prod_{k\neq i,j}(\gamma^2 + \lambda_i\lambda_k)(\gamma^2 + \lambda_j\lambda_k)}{\gamma^8 \prod_{k\neq i}(\lambda_i - \lambda_k)\prod_{k\neq j}(\lambda_j - \lambda_k)} \\
&= \frac{(\gamma\lambda_i^{-1} + 1 + \gamma^{-1}\lambda_i)(\gamma\lambda_j^{-1} + 1 + \gamma^{-1}\lambda_j)}{(1 - \lambda_j\lambda_i^{-1})(1 - \lambda_i\lambda_j^{-1})} \; \frac{\prod_{k\neq i,j}(\gamma^2 + \lambda_i\lambda_k)(\gamma^2 + \lambda_j\lambda_k)}{\gamma^6 \prod_{k\neq i,j}(\lambda_i - \lambda_k)(\lambda_j - \lambda_k)}.
\end{aligned}$$

The first of the two quotients is easily seen to be real. For the second quotient,

$$\overline{\left(\frac{\prod_{k\neq i,j}(\gamma^2 + \lambda_i\lambda_k)(\gamma^2 + \lambda_j\lambda_k)}{\gamma^6 \prod_{k\neq i,j}(\lambda_i - \lambda_k)(\lambda_j - \lambda_k)}\right)} = \frac{\prod_{k\neq i,j}(\gamma^{-2} + \lambda_i^{-1}\lambda_k^{-1})(\gamma^{-2} + \lambda_j^{-1}\lambda_k^{-1})}{\gamma^{-6} \prod_{k\neq i,j}(\lambda_i^{-1} - \lambda_k^{-1})(\lambda_j^{-1} - \lambda_k^{-1})}.$$

Multiply the numerator and the denominator by $\gamma^{12}\lambda_i^3\lambda_j^3 \prod_{k\neq i,j} \lambda_k^2$ to see that this is still

$$\frac{\prod_{k\neq i,j}(\gamma^2 + \lambda_i\lambda_k)(\gamma^2 + \lambda_j\lambda_k)}{\gamma^6 \prod_{k\neq i,j}(\lambda_i - \lambda_k)(\lambda_j - \lambda_k)}.$$

For the case $i = j$, note that $\sum_{k=1}^d e_{A,k} = I$, so

$$\begin{aligned}
e_{B,i} &= e_{B,i} I e_{B,i} \\
&= e_{B,i} \sum_{k=1}^d e_{A,k} e_{B,i} \\
&= \sum_{k=1}^d e_{B,i} e_{A,k} e_{B,i} \\
&= \sum_{k=1}^d \mu_{ik}^{(d)} e_{B,i}.
\end{aligned}$$

Hence $\sum_{k=1}^d \mu_{ik}^{(d)} = 1$, and $\mu_{ii}^{(d)} = 1 - \sum_{k\neq i} \mu_{ik}^{(d)}$ is real. $\square$

Proposition 2.2. *$S = \{e_{A,i}e_{B,1}e_{A,j} \mid 1 \le i,j \le d,\ i \ne j\} \cup \{e_{A,i} \mid 1 \le i \le d\}$ is a basis for the $\mathbb{C}$-vector space $\mathcal{B}$.*

Proof. Suppose
$$\sum_{i=1}^{d}\sum_{\substack{j=1\\ j\ne i}}^{d}\alpha_{ij}e_{A,i}e_{B,1}e_{A,j} + \sum_{i=1}^{d}\alpha_{ii}e_{A,i} = 0.$$
Multiply by $e_{A,i}$ both on the left and on the right. The only term of the sum that survives is
$$\alpha_{ii}e_{A,i} = 0.$$
Let v_i be an eigenvector of A corresponding to λ_i. Then $e_{A,i}v_i = v_i \ne 0$, so $e_{A,i} \ne 0$. Hence $\alpha_{ii} = 0$.

For $i \ne j$, multiplying by $e_{A,i}$ on the left and by $e_{A,j}$ on the right shows
$$\alpha_{ij}e_{A,i}e_{B,1}e_{A,j} = 0.$$
But
$$e_{B,1}e_{A,i}e_{B,1}e_{A,j}e_{B,1} = (e_{B,1}e_{A,i}e_{B,1})(e_{B,1}e_{A,j}e_{B,1}) = \mu_{1j}^{(d)}\mu_{1i}^{(d)}e_{B,1} \ne 0,$$
so $e_{A,i}e_{B,1}e_{A,j} \ne 0$. Hence $\alpha_{ij} = 0$. So S is linearly independent. It has d^2 elements, hence it is a basis of the d^2-dimensional space $\mathcal{B}$. □

Note: if we know $\mu_{ii}^{(d)} \ne 0$ for all i, we can use the basis $S' = \{e_{A,i}e_{B,1}e_{A,j} \mid 1 \le i,j \le d\}$ instead of S. As $e_{A,i}e_{B,1}e_{A,i} = \mu_{ii}^{(d)}e_{A,i}$, S' is almost the same as S. Since S' is more symmetric than S, its use makes the following computations simpler and the arguments more transparent. In the most general case however, $\mu_{ii}^{(d)}$ could be 0.

Define $\imath : \mathbb{C} \to \mathbb{C}$ as the usual complex conjugation. Extend $\imath$ to $\mathcal{B} \to \mathcal{B}$ by requiring $\imath$ to be an antilinear map with $\imath(e_{A,i}) = e_{A,i}$ and $\imath(e_{A,i}e_{B,1}e_{A,j}) = e_{A,j}e_{B,1}e_{A,i}$ for $i \ne j$. Note that $\imath(\mu_{ij}^{(d)}) = \mu_{ij}^{(d)}$.

Lemma 2.3. *$\imath$ as defined above is an antihomomorphism on the algebra $\mathcal{B}$ and $\imath^2 = \mathrm{Id}_{\mathcal{B}}$.*

Proof. It is sufficient to prove that $\imath$ acts as an antihomomorphism on the elements of the basis S. S has two different types of elements, therefore we will have four different cases. Since each can be verified directly by a simple computation, we will show the details for only one:

(1)
$$\imath(e_{A,i}e_{A,j}) = \imath(e_{A,j})\imath(e_{A,i}).$$

(2)
$$\begin{aligned}\imath(e_{A,i}(e_{A,j}e_{B,1}e_{A,k})) &= \imath(e_{A,j}e_{B,1}e_{A,k})\imath(e_{A,i}),\\ \imath((e_{A,i}e_{B,1}e_{A,j})e_{A,k}) &= \imath(e_{A,k})\imath(e_{A,j}e_{B,1}e_{A,k}).\end{aligned}$$

(3) For $i \ne k$,
$$\imath((e_{A,i}e_{B,1}e_{A,j})(e_{A,k}e_{B,1}e_{A,l})) = (e_{A,l}e_{B,1}e_{A,k})(e_{A,j}e_{B,1}e_{A,i}).$$

(4)

$$\begin{aligned}
\imath((e_{A,i}e_{B,1}e_{A,j})(e_{A,j}e_{B,1}e_{A,k})) &= \imath(e_{A,i}(e_{B,1}e_{A,j}e_{B,1})e_{A,k}) \\
&= \imath(e_{A,i}(\mu_{1j}^{(d)}e_{B,1})e_{A,k}) \\
&= \overline{\mu_{1j}^{(d)}}\imath(e_{A,i}e_{B,1}e_{A,k}) \\
&= \mu_{1j}^{(d)}e_{A,k}e_{B,1}e_{A,i}.
\end{aligned}$$

Also

$$\begin{aligned}
\imath(e_{A,j}e_{B,1}e_{A,k})\imath(e_{A,i}e_{B,1}e_{A,j}) &= (e_{A,k}e_{B,1}e_{A,j})(e_{A,j}e_{B,1}e_{A,i}) \\
&= e_{A,k}(e_{B,1}e_{A,j}e_{B,1})e_{A,i} \\
&= \mu_{1j}^{(d)}e_{A,k}e_{B,1}e_{A,i}.
\end{aligned}$$

That $\imath^2 = \mathrm{Id}_{\mathcal{B}}$ follows immediately from the definition. □

Lemma 2.4. $\imath(e_{B,1}) = e_{B,1}$.

Proof. First note that $\imath(e_{A,i}e_{B,1}e_{A,i}) = \imath(\mu_{ii}^{(d)}e_{A,i}) = \mu_{ii}^{(d)}e_{A,i} = e_{A,i}e_{B,1}e_{A,i}$. Multiply $e_{B,1}$ by $1 = \sum_{i=1}^{d} e_{A,i}$ on both sides:

$$e_{B,1} = \left(\sum_{i=1}^{d} e_{A,i}\right) e_{B,1} \left(\sum_{j=1}^{d} e_{A,j}\right) = \sum_{i,j} e_{A,i}e_{B,1}e_{A,j}$$

into

$$\begin{aligned}
\imath(e_{B,1}) &= \imath\left(\sum_{i=1}^{d}\sum_{j=1}^{d} e_{A,i}e_{B,1}e_{A,j}\right) = \sum_{i=1}^{d}\sum_{j=1}^{d}\imath(e_{A,i}e_{B,1}e_{A,j}) \\
&= \sum_{i=1}^{d}\sum_{j=1}^{d}(e_{A,j}e_{B,1}e_{A,i}) = e_{B,1}.
\end{aligned}$$

□

Corollary 2.5. $\imath(A) = A^{-1}$ *and* $\imath(I) = I$.

Proof.

$$\imath(A) = \imath(\sum_{i=1}^{d}\lambda_i e_{A,i}) = \sum_{i=1}^{d}\overline{\lambda_i}\imath(e_{A,i}) = \sum_{i=1}^{d}\lambda_i^{-1}e_{A,i} = A^{-1}.$$

Similarly,

$$\imath(I) = \imath(\sum_{i=1}^{d} e_{A,i}) = \sum_{i=1}^{d}\imath(e_{A,i}) = \sum_{i=1}^{d} e_{A,i} = I.$$

□

Lemma 2.6. $\imath(B) = B^{-1}$.

Proof. Note that

$$A^{-1}\imath(B)A^{-1} = \imath(A)\imath(B)\imath(A) = \imath(ABA) = \imath(BAB) = \imath(B)A^{-1}\imath(B).$$

That is, A^{-1} and $\imath(B)$ satisfy the braid relation. So the group homomorphism $\rho' : B_3 \to \mathrm{GL}(V)$ defined by $\rho'(\sigma_1) = A^{-1}$ and $\rho'(\sigma_2) = \imath(B)$ is another representation of B_3 on V. Once again, the braid relation implies that A^{-1} and $\imath(B)$ are conjugates. Hence they have the same eigenvalues, namely $\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_d^{-1}$.

But $\imath : \mathcal{B} \to \mathcal{B}$ only permutes the basis S of $\mathcal{B} = \mathrm{End}(V)$. Hence $\imath(\mathcal{B}) = \imath(\mathrm{End}(V)) = \mathrm{End}(V)$ and A^{-1} and $\imath(B)$ generate the algebra $\mathrm{End}(V)$. That is, ρ' is also a simple representation of B_3.

Now, $(A^{-1}\imath(B))^3 = \imath(BA)^3 = \imath(AB)^3 = \imath(\delta I) = \overline{\delta} = \delta^{-1}I$ (recall $|\delta| = 1$). By Corollary 1.1, the eigenvalues $\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_d^{-1}$ (if $d = 2, 3$) or the eigenvalues together with δ (if $d = 4, 5$) uniquely determine a simple representation of B_3 on V up to isomorphism.

But we already know such a representation, namely $\sigma_1 \mapsto A^{-1}$ and $\sigma_2 \mapsto B^{-1}$. Hence there exists $M \in \mathrm{GL}(V)$ such that $A^{-1} = MA^{-1}M^{-1}$ and $\imath(B) = MB^{-1}M^{-1}$. Then M is in the centralizer of A.

$$\begin{aligned}
Me_{B,1}M^{-1} &= M\left(\prod_{i=2}^{d} \frac{B - \lambda_i}{\lambda_1 - \lambda_i}\right) M^{-1} \\
&= \prod_{i=2}^{d} \frac{MBM^{-1} - \lambda_i}{\lambda_1 - \lambda_i} \\
&= \prod_{i=2}^{d} \frac{\imath(B^{-1}) - \lambda_i}{\lambda_1 - \lambda_i} \\
&= \prod_{i=2}^{d} \imath\left(\frac{B^{-1} - \lambda_i^{-1}}{\lambda_1^{-1} - \lambda_i^{-1}}\right) \\
&= \imath\left(\prod_{i=2}^{d} \frac{B^{-1} - \lambda_i^{-1}}{\lambda_1^{-1} - \lambda_i^{-1}}\right).
\end{aligned}$$

Call the quantity in parentheses ϕ. Note that ϕ is the eigenprojection to the subspace spanned by the eigenvector w_1 of B^{-1} with eigenvalue λ_1^{-1}. But the eigenvectors $w_1, w_2, \ldots, w_d$ of B^{-1} are also eigenvectors of B and span V (the eigenvalues are distinct). Hence $\phi(w_1) = w_1 = e_{B,1}w_1$ and $\phi(w_i) = 0 = e_{B,1}w_i$ for $i \geq 2$. That is, $\phi = e_{B,1}$ as their action on the basis $\{w_1, w_2, \ldots, w_d\}$ is identical. Then Lemma 2.4 shows $\imath(Me_{B,1}M^{-1}) = \imath(\phi) = \imath(e_{B,1}) = e_{B,1}$.

Hence conjugation by M is a $\mathcal{B}$-algebra isomorphism that fixes A and $e_{B,1}$. But A and $e_{B,1}$ generate the basis S of $\mathcal{B}$, hence they generate the algebra $\mathcal{B}$. So conjugation by M must fix every element of $\mathcal{B}$. In particular, $\imath(B) = MB^{-1}M^{-1} = B^{-1}$. $\square$

3. An invariant inner-product

Let $\mathcal{B}$ act on the left algebra ideal $\mathcal{B}e_{B,1}$. Note that $\mathcal{B}e_{B,1}$ is a d-dimensional $\mathbb{C}$-vector space, as $e_{B,1}$ is an idempotent of rank 1.

Definition 3.1. Define the form $\langle .,. \rangle$ on $\mathcal{B}e_{B,1}$ by

$$\langle ae_{B,1}, be_{B,1}\rangle\, e_{B,1} = \imath(be_{B,1})ae_{B,1} = e_{B,1}\imath(b)ae_{B,1}$$

for $ae_{B,1}, be_{B,1} \in \mathcal{B}e_{B,1}$.

It is easy to verify that $\langle . , . \rangle$ is a sesquilinear form on the $\mathbb{C}$-vector space $\mathcal{B}e_{B,1}$. Since $\imath(A) = A^{-1}$ and $\imath(B) = B^{-1}$, this form is clearly invariant under the action by A and B; hence $\rho(B_3)$.

Lemma 3.2. *$T = \{e_{A,i}e_{B,1} \mid 2 \leq i \leq d\} \cup \{ABAe_{B,1}\}$ is a basis for left algebra ideal $\mathcal{B}e_{B,1}$ considered as a $\mathbb{C}$-vector space.*

Proof. Suppose

$$\alpha_1 ABAe_{B,1} + \sum_{i=2}^{d} \alpha_i e_{A,i}e_{B,1} = 0.$$

Note that $(e_{A,i}ABAe_{B,1})(ABA)^{-1} = e_{A,i}e_{A,1} = \delta_{1i}$. Since $(ABA)^{-1}$ is invertible $e_{A,i}ABAe_{B,1} = 0$ if and only if $i \geq 2$.

Multiply by $e_{A,1}$ on the left. Then $\alpha_1 e_{A,1}ABAe_{B,1} = 0$. But $e_{A,1}ABAe_{B,1} \neq 0$, so $\alpha_1 = 0$.

Now, multiply by $e_{A,i}$ $(i \geq 2)$ on the left. Then $\alpha_i e_{A,i}e_{B,1} = 0$. We know $e_{B,1}e_{A,i}e_{B,1} = \mu_{1i}^{(d)} e_{B,1} \neq 0$ by simplicity, so $e_{A,i}e_{B,1} \neq 0$ and $\alpha_i = 0$.

Hence T is a linearly independent set, and we can conclude that it is a basis of the d-dimensional vector space $\mathcal{B}e_{B,1}$. □

Note: if we know $e_{A,1}e_{B,1} \neq 0$, we can use the more symmetric basis $T' = \{e_{A,i}e_{B,1} \mid 1 \leq i \leq d\}$ to simplify this and some of the following computations. Unfortunately, $e_{A,1}e_{B,1}$ could in general be 0. In particular, if $\mu_{11}^{(d)} = 0$, then $e_{A,1}e_{B,1} = 0$ too.

Theorem 3.3. *The braid representation $\mathcal{B}$ is unitarizable if and only if $\mu_{1i}^{(d)} > 0$ for all $2 \leq i \leq d$.*

Proof. Suppose $\mu_{1i}^{(d)} > 0$ for all $2 \leq i \leq d$. Consider the action of $\mathcal{B}$ on $\mathcal{B}e_{B,1}$. The sesquilinear form defined above is invariant under the action of $\rho(B_3)$. So it is sufficient to show that it is an inner product. That is, we need to prove that it is positive definite. On the basis T:

$$\begin{aligned}
\langle e_{A,i}e_{B,1}, e_{A,i}e_{B,1} \rangle \, e_{B,1} &= e_{B,1}\imath(e_{A,i})e_{A,i}e_{B,1} = e_{B,1}e_{A,i}e_{A,i}e_{B,1} \\
&= e_{B,1}e_{A,i}e_{B,1} = \mu_{1i}^{(d)} e_{B,1}, \\
\langle ABAe_{B,1}, ABAe_{B,1} \rangle \, e_{B,1} &= \langle e_{B,1}, e_{B,1} \rangle \, e_{B,1} = e_{B,1}e_{B,1} = e_{B,1}.
\end{aligned}$$

Hence $\langle e_{A,i}e_{B,1}, e_{A,i}e_{B,1} \rangle = \mu_{1i}^{(d)} > 0$ for $i \geq 2$ by assumption, and

$$\langle ABAe_{B,1},\ ABAe_{B,1} \rangle = 1.$$

We claim that T is orthogonal with respect to $\langle ., , \rangle$. Let $i, j \neq 1$ and $i \neq j$:

$$\begin{aligned}
\langle e_{A,i}e_{B,1}, e_{A,j}e_{B,1} \rangle \, e_{B,1} &= e_{B,1}\imath(e_{A,i})e_{A,j}e_{B,1} = e_{B,1}e_{A,i}e_{A,j}e_{B,1} = 0, \\
\langle ABAe_{B,1}, e_{A,i}e_{B,1} \rangle \, e_{B,1} &= e_{B,1}\imath(e_{A,i})ABAe_{B,1} = e_{B,1}e_{A,i}ABAe_{B,1} = 0.
\end{aligned}$$

We used $e_{A,i}ABAe_{B,1} = 0$ in the last computation as before in Lemma 3.2.

Hence $\langle ., . \rangle$ is a positive definite form. Then $\mathcal{B}e_{B,1}$ is a $\mathbb{C}$-vector space with inner product $\langle . , . \rangle$ and the action of $\rho(B_3)$ on this space is unitary.

Conversely, suppose $\mathcal{B}$ is unitarizable. So there exists V a $\mathbb{C}$-vector space with inner product $\langle . , . \rangle$ and $\rho : B_3 \to \mathrm{GL}(V)$ such that $A = \rho(\sigma_1)$ and $B = \rho(\sigma_2)$ act as unitary operators on V. Let * be the transpose induced by $\langle ., . \rangle$. We know that

$A^* = A^{-1}$ and $B^* = B^{-1}$. Let $v \in V$ be an eigenvector of B with eigenvalue λ_1. Then $e_{B,1}v = v$ and

$$\begin{aligned} 0 \le \langle e_{A,i}e_{B,1}v, e_{A,i}e_{B,1}v\rangle &= \left\langle v, e_{B,1}^* e_{A,i}^* e_{A,i}e_{B,1}v\right\rangle \\ &= \langle v, e_{B,1}e_{A,i}e_{B,1}v\rangle = \left\langle v, \mu_{1i}^{(d)} e_{B,1}v\right\rangle = \mu_{1i}^{(d)}\langle v, v\rangle . \end{aligned}$$

Hence $\mu_{1i}^{(d)} \ge 0$. We know $\mu_{1i}^{(d)} \ne 0$ for $i \ge 2$ by simplicity, so $\mu_{1i}^{(d)} > 0$ in this case. □

4. Examples

Example 4.1. $d = 2$:

$$\begin{aligned} \mu_{12}^{(2)} &= \frac{-\lambda_1^2 + \lambda_1\lambda_2 - \lambda_2^2}{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_1)} \\ &= \frac{\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2}{(\lambda_1 - \lambda_2)^2} \\ &= 1 + \frac{\lambda_1\lambda_2}{(\lambda_1 - \lambda_2)^2} \\ &= 1 - \frac{1}{(\lambda_1/\lambda_2 - 1)(\lambda_2/\lambda_1 - 1)} \\ &= 1 - \left|\frac{\lambda_1}{\lambda_2} - 1\right|^{-2} > 0. \end{aligned}$$

That is,

$$\left|\frac{\lambda_1}{\lambda_2} - 1\right| > 1,$$

or $\lambda_1/\lambda_2 = e^{it}$ for $\pi/3 < t < 5\pi/3$.

Example 4.2. $d = 3$:

$$\begin{aligned} \mu_{12}^{(3)} &= \frac{(\lambda_1^2 + \lambda_2\lambda_3)(\lambda_2^2 + \lambda_1\lambda_3)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \\ &= \frac{\left(1 + \frac{\lambda_3}{\lambda_1}\frac{\lambda_2}{\lambda_1}\right)\left(\frac{\lambda_2}{\lambda_1} + \frac{\lambda_3}{\lambda_2}\right)}{\left(1 - \frac{\lambda_2}{\lambda_1}\right)\left(1 - \frac{\lambda_1}{\lambda_2}\right)\left(1 - \frac{\lambda_3}{\lambda_1}\right)\left(\frac{\lambda_2}{\lambda_1} - \frac{\lambda_3}{\lambda_1}\right)}, \\ \mu_{13}^{(3)} &= \frac{(\lambda_1^2 + \lambda_2\lambda_3)(\lambda_3^2 + \lambda_1\lambda_2)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \\ &= \frac{\left(1 + \frac{\lambda_2}{\lambda_1}\frac{\lambda_3}{\lambda_1}\right)\left(\frac{\lambda_3}{\lambda_1} + \frac{\lambda_2}{\lambda_3}\right)}{\left(1 - \frac{\lambda_3}{\lambda_1}\right)\left(1 - \frac{\lambda_1}{\lambda_3}\right)\left(1 - \frac{\lambda_2}{\lambda_1}\right)\left(\frac{\lambda_3}{\lambda_1} - \frac{\lambda_2}{\lambda_1}\right)}. \end{aligned}$$

Let $\omega_2 = \lambda_2/\lambda_1$ and $\omega_3 = \lambda_3/\lambda_1$. Then

$$\begin{aligned}
\mu_{12}^{(3)} &= \frac{(1+\omega_3\omega_2)\left(\omega_2+\omega_3\omega_2^{-1}\right)}{|1-\omega_2|^2\,(1-\omega_3)\,(\omega_2-\omega_3)},\\
\mu_{13}^{(3)} &= \frac{(1+\omega_2\omega_3)\left(\omega_3+\omega_2\omega_3^{-1}\right)}{|1-\omega_3|^2\,(1-\omega_2)\,(\omega_3-\omega_2)}.
\end{aligned}$$

Let $e^{2\pi t_2} = \omega_2$ and $e^{2\pi t_3} = \omega_3$. So we are looking for $(t_2, t_3) \in [0,1)^2$ such that both $\mu_{12}^{(3)} > 0$ and $\mu_{13}^{(3)} > 0$. $\mu_{12}^{(3)}$ and $\mu_{13}^{(3)}$ can change signs at

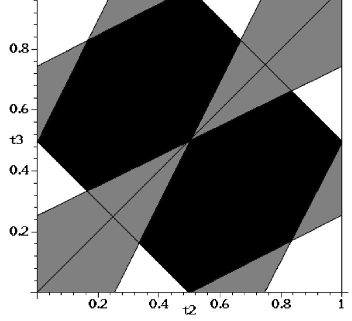
$$\begin{aligned}
\omega_2\omega_3 &= -1,\\
\omega_3\omega_2^{-1} &= -\omega_2,\\
\omega_2\omega_3^{-1} &= -\omega_3,\\
w_2 &= 1,\\
w_3 &= 1,\\
w_2 &= w_3.
\end{aligned}$$

These equations can be transformed into linear equations in t_2 and t_3 by taking logs:

$$\begin{aligned}
t_2 + t_3 &= \frac{1}{2},\\
t_3 &= 2t_2 + \frac{1}{2},\\
t_2 &= 2t_3 + \frac{1}{2},\\
t_2 &= 0,\\
t_3 &= 0,\\
t_2 &= t_3.
\end{aligned}$$

Of course, the above equations are all understood mod 1.

Computation by Maple shows that $\mu_{12}^{(3)} > 0$ and $\mu_{13}^{(3)} > 0$ in the open set colored black on the plot below. The grey regions are those where one of $\mu_{12}^{(3)}$ and $\mu_{13}^{(3)}$ is positive and the other is negative. The line $t_2 = t_3$ corresponds to $\lambda_2 = \lambda_3$, in which case the representation cannot be unitarizable.



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